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The aleph partial derivative

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Abstract

In this paper we define the "aleph" partial derivative through an adequate generalization of the usual concept of finite increment. The main properties of this operator are also established. This derivative extends the classical partial derivative in the same way that the "aleph" derivative (Aldanondo [1]) generalizes the familiar derivative of an arbitrary function.

Key words: Partial derivative, aleph.

La derivada parcial aleph

Resumen

En este trabajo se define la derivada parcial "aleph" con el uso de una adecuada generalización del concepto usual de incremento finito. Se establecen las propiedades más importantes de este operador y se comprueba cómo esta nueva derivada extiende la derivada parcial clásica, de igual manera que la derivada "aleph" (Aldanondo [1]) generaliza la derivada familiar de una función arbitraria.

Palabras claves: Derivada parcial, aleph.

Introduction

The aleph derivative of y(t) is written $\alpha[y(t)]$ and is a generalization of the familiar derivative dy/dt. It was introduced by I. Aldanondo [1] and defined as follows:

$$\alpha[y(t)] = [D(y(t))/g(t)y(t)]/f(t) \qquad (1)$$

where y(t), f(t) and g(t) are real-valued continuosly differentiable functions on an interval $I = [t_0,a]$, $f(t) = 0$ ($\supset t \in I$) and $D = d/dt$.

The expression (1) is said to be the "aleph derivative" of y(t) with respect to f(t) and g(t), and f and g are called generating functions of the operator $\alpha = (D - g)/f$.

In the present paper we define the "aleph partial derivative" which generalizes the classical partial derivative, in the same way as that used by I. Aldanondo [1], here considering another adequate extension of the usual concept of finite increment. Furthermore, the main properties of this aleph partial derivative are established.

The "aleph" partial derivative

Let $z = z(x)$ be a continuosly differentiable function in a domain $D \subset \mathbf{R}^n$. Let also $f_1(x,h)$ and $g_1(x,h)$ be two functions of the class C^1 in the set $D \times I$, I being an interval of $\mathbf{R}$ and $h \in I$. Define the following functions:

$$f_2(x) = \lim_{h\to 0} \frac{f_1(x,h) - 1}{h} \quad g_2(x) = \lim_{h\to 0} \frac{g_1(x,h) - 1}{h} \qquad (2)$$

i.e., when the parameter is small enough ($|h|<\delta$, $\delta>0$), one has for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in D:

$f_1(x,h) \cong 1 + hf_2(x), \quad g_1(x,h) \cong 1 + hg_2(x).$

The expression:

$$\Delta_{a_i} z(x) = z(x_1,x_2,\ldots,x_i+h,\ldots,x_n) - g_1(x,h)\, z(x) \quad (3)$$

is said the generalized "aleph" increment of the function z(x) with respect to $x_i \in \mathbf{R}$ (i = 1,2,...,n) corresponding to $g_1(x)$. Same denomination is valid for the expression which corresponds to $f_1(x)$.

Now, we define the "aleph increment quotient" of z(x) with respect the variable x_i, by means of the following expression:

$$\frac{\Delta_{a_i} z(x)}{\Delta_{a_i} x_i} = \frac{z(x_1,x_2,\ldots,x_{i-1},x_i+h,x_{i+1},\ldots,x_n) - g_1(x,h)\, z(x)}{x_i + h - f_1(x,h)\, x_i} \quad (4)$$

Definition 1

The limit, if it exists:

$$\lim_{h\to 0} \frac{\Delta_{a_i} z(x)}{\Delta_{a_i} x_i} = \lim_{h\to 0} \frac{z(x_1,x_2,\ldots,x_i+h,\ldots,x_n) - z(x) - h\, g_2(x)\, z(x)}{x_i + h - f_2(x)\, x_i} =$$

$$= \lim_{h\to 0} \frac{z(x_1,x_2,\ldots,x_i+h,\ldots,x_n) - z(x)}{h[1 - x_i f_2(x)]} - g_2(x)\, z(x) \frac{1}{1 - x_i f_2(x)} =$$

$$\frac{\partial z/\partial x_i - g_2(x)\, z(x)}{1 - x_i f_2(x)}$$

which will be denoted by $\partial_{a_i}(z)$, is said to be the "aleph partial derivative" of z(x) with respect to the variable x_i. Putting $1 - x_i f_2(x) = f_i(x)$, we can write:

$$\partial_{a_i}(z) = \frac{\partial z/\partial x_i - g_2(x)\, z(x)}{f_i(x)} \quad (5)$$

whenever $f_i(x) \neq 0$, $\forall\, x \in D$.

The functions f_i and g_2 are denominated generating functions of ∂_{a_i}. This operator ∂_{a_i} is fully determined by those functions f and g, and we will write:

$$\partial_{a_i} \equiv [f_i, g_2] \quad (6)$$

Remark 1

Clearly, when in (5) it happens that $g_2(x) = 0$, $f_i(x) = 1$ ($\forall x \in D$), it is obtained that $\partial_{a_i}(z)$ coincides with $\partial z/\partial x_i$, i.e., the aleph partial derivative represents an operator more general that the usual partial derivative operator.

Proposition 1

"The aleph partial derivative is a linear operator from the space C^1 (D) into itself"

Proof: For $z,w \in C^1$ (D), and $\alpha,\beta \in \mathbf{R}$ we can easily to prove that:

$$\partial_{a_i}[\alpha z + \beta w] = [\alpha\, \partial z/\partial x_i + \beta\, \partial w/\partial x_i - g_2(x)\,(\alpha z + \beta w)] / f_i(x) =$$

$$= \alpha \partial_{a_i}(z) + \beta \partial_{a_i}(w)$$

Remark 2

Observe that, in general, the "aleph" partial derivative of a constant function is not zero, since:

$$\partial_{a_i}(c) = -c \frac{g_2(x)}{f_i(x)}, \ \forall c \in \mathbf{R}$$

It is important to know when two given functions $f_2(x)$ and $g_2(x)$ in C^1 (D) can be used to define generating functions for the operator ∂_{a_i}. Hence we must study the kernel of ∂_{a_i}.

Let V(x) be a function of C^1 (D). If V(x) belongs to Ker ∂_{a_i}, we can write:

$$\frac{\partial V/\partial x_i - g_2(x) V(x)}{f_i(x)} = 0, \ \forall\, x \in D;\ i.e.$$

$$\partial V/\partial x_i(x) - g_2(x) V(x) = 0$$

which is a linear partial differential equation of the first order, which admits as general solution:

$$V(x) = \alpha(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x)\, e^{\int g_2(x)\, dx_i} \quad (7)$$

α being a function in $\mathbf{R}^{n-1}$ not depending of the variable x_i.

Conversely, it is easily obtained that any function V(x) given by means of (7), belongs to Ker ∂_{a_i}. Hence the following theorem holds.

Theorem 1

"The kernel of the aleph operator ∂_{a_i} is:

$$\text{Ker } \partial_{a_i} = \{ \alpha(\bar{x})\, e^{G(x)} \,/\, x \in D \}$$

α and G being C^1-functions, with α depending to the variable $\bar{x} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and G to the variable x, and such that $\partial G/\partial x_i = g_2$, where g_2 is the generating function".

On the other hand, and following with our research of the structure of the operator ∂_{a_i}, consider the set $F = \{z(x) \in C^1(D) \,/\, \text{there exists: } \partial_{a_i}(z)\}$. Let z_0 be a fixed function of F. We say that two functions u and v of F are equivalent with respect to z_0, if and only if, there exists a function $m \in F$ such that:

$$u = v + m\, z_0$$

We will denote this relation by $\approx$. It can proved without difficulty that $\approx$ is an equivalence relation. {z} will be the equivalence class of z and F/z_0 the quotient set. In this set define the following operator:

$$\partial_{a_i}[\{z\}] = \{ \partial_{a_i}(z) \}$$

Now, we need prove that ∂_{a_i} satisfies the uniform property, i.e. the class of this operator does not depend of the chosen function z_0.

Let $z, u \in F$ with $z, u \in \{z\}$. Then there exists a function $m \in F$ such that $u = z + mz_0$. For applying ∂_{a_i} one has:

$$\partial_{a_i}(\{u\}) = \{ \partial_{a_i}(u) \}, \text{ i.e.}$$

$$\partial_{a_i}(z) + m\partial_{a_i}(z_0) = \partial_{a_i}(z) + nz_0, \text{ for } n \in F$$

and consequently:

$$m\partial_{a_i}(z_0) = n\, z_0$$

or:

$$\frac{(\partial z_0(x)/\partial x_i) - g_2(x)\, z_0(x)}{f_i(x)} = \frac{n(x)}{m(x)}\, z_0(x)$$

Thus, by virtue of the theorem 1, $z_0(x)$ must be of the form:

$$z_0(x) = \alpha(\bar{x})\, e^{\int [G(x) + \lambda F(x)]\, dx_i}$$

where: $\bar{x} \in \mathbf{R}^{n-1}$, $\alpha \in C^1(\mathbf{R}^{n-1})$, $\lambda \in C^1(D)$, G and $F \in C^1(D)$, with:

$$\partial G(x)/\partial x_i = g_2(x) \text{ and } \partial F(x)/\partial x_i = f_i(x),\ \forall\, x \in D$$

Hence, the uniform property will be verified in a quotient set of the form $F/(G+\lambda F)$. In the particular case in which $\lambda(x) = 0$, $\forall\, x \in D$, the operator ∂_{a_i} satisfies this property in the space $F/\text{Ker } \partial_{a_i}$.

Properties of the aleph partial derivative

In this section we prove some basic properties of the "aleph" partial operator ∂_{a_i} (i = 1,2, ..., n). Henceforth, we shall denote for each i = 1,2, ..., n, f_i and g_i as the generating functions of the operator ∂_{a_i}.

1. Firstly, we define the second order aleph operator by means of the following representation:

$$\partial_{a_{ij}}(z) = \partial_{a_i}[\partial_{a_j}(z)] \quad (8)$$

z being a function of the class C^2 in D, and i,j = 1,2, ..., n.

The next result is fundamental for many of the applications of the operator ∂_{a_i}.

Formerly, it is necessary to give the conditions that must verify the generating functions of the operators according to the equality

$$\partial_{a_{ij}}(z) = \partial_{a_{ji}}(z) \qquad (9)$$

be valid for any function z l F, provided that $\partial_{a_i}(z)$ and $\partial_{a_j}(z)$ also belongs to F. By applying (8) two consecutive times and taking into account the definition of the operator $\partial_{a_{ij}}$, we get:

$$\partial_{a_{ij}}(z) = \partial_{a_i}[\partial_{a_j}(z)] = \partial_{a_i}\left[\frac{\partial z/\partial x_j - g_j z}{f_j}\right] =$$
$$= \frac{1}{f_i}\left[\frac{\partial}{\partial x_i}\left[\frac{\partial z/\partial x_j - g_j z}{f_j}\right] - g_i\left[\frac{\partial z/\partial x_j - g_j z}{f_j}\right]\right] =$$
$$= \frac{1}{f_i f_j}\left[\frac{\partial^2 z}{\partial x_i \partial x_j} - g_j\frac{\partial z}{\partial x_i} - \left[\frac{\partial f_j}{\partial x_i}\frac{1}{f_j} + g_i\right]\frac{\partial z}{\partial x_j} - \left[\frac{\partial g_j}{\partial x_i} - \frac{g_j}{f_j}\frac{\partial f_j}{\partial x_i} - g_i g_j\right] z\right]$$

and in the same way for $\partial_{a_{ji}}$:

$$\partial_{a_{ji}}(z) = \frac{1}{f_j f_i}\left[\frac{\partial^2 z}{\partial x_j \partial x_i} - g_i\frac{\partial z}{\partial x_j} - \left[\frac{\partial f_i}{\partial x_j}\frac{1}{f_i} + g_j\right]\frac{\partial z}{\partial x_i} - \left[\frac{\partial g_i}{\partial x_j} - \frac{g_i}{f_i}\frac{\partial f_i}{\partial x_j} - g_j g_i\right] z\right]$$

Now, the equality of both expressions allow to conclude that:

$$\frac{\partial f_i}{\partial x_j}\frac{1}{f_i} = 0,$$

i.e. f_i must be a function of the variable

$$\bar{x} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$\frac{\partial f_j}{\partial x_i}\frac{1}{f_j} = 0,$$

i.e. f_j must be a function of

$$\bar{\bar{x}} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Furthermore, it is necessary that:

$$\frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j}$$

Hence, we can state the following theorem:

Theorem 2

"If z is a function in $C^2(D)$ such that z, $\partial_{a_i}(z)$ and $\partial_{a_j}(z)$ belong to the set F, the following relation

$$\partial_{a_i}[\partial_{a_j}(z)] = \partial_{a_j}[\partial_{a_i}(z)]$$

holds, if and only if, the generating functions f_i, g_i of ∂_{a_i} and the f_j, g_j ones of ∂_{a_j} belong to $C^2(D)$ and also satisfy:

$$f_i(x) = f(\bar{x}),\ f_j(x) = f(\bar{\bar{x}})$$

Moreover, there exists a function G belonging to $C^1(D)$ for which:

$$g_i(x) = (\nabla G)_i(x) \text{ and } g_j(x) = (\nabla G)_j(x)\text{"}$$

2. The aleph partial derivative of a product of functions.

If z and w are two functions which belong to F, we can obtain three different expressions for $\partial_{a_i}(z.w)$, namely

$$\text{i) } \partial_{a_i}(z.w) = \frac{\partial(z.w)/\partial x_i - g_i.z.w}{f_i} = \frac{\partial z/\partial x_i - g_i z}{f_i} w + \frac{z}{f_i}\frac{\partial w}{\partial x_i} =$$
$$= w.\partial_{a_i}(z) + \frac{z}{f_i}\frac{\partial w}{\partial x_i}$$

Now, by using the following another form to operate, it results:

$$\text{ii) } \partial_{a_i}(z.w) = z.\partial_{a_i}(w) + \frac{w}{f_i}\frac{\partial z}{\partial x_i}.$$

Finally, it can also be found:

iii) $\partial_{a_i}(z.w) = [\partial z/\partial x_i - g_i z].w/f_i + [\partial w/\partial x_i - g_i w].z/f_i +$
$+ z.w.g_i/f_i = z.\partial_{a_i}(w) + w.\partial_{a_i}(z) + z.w.g_i/f_i.$

3. The aleph partial derivative of a quotient of two functions.

In the same way, for z, w $\in$ F, w(x) $\neq$ 0, $\forall$ x $\in$ D it can be inferred that:

$$\partial_{a_i}(z/w) = \partial_{a_i}(z)/w - \frac{z}{w^2}\frac{1}{f_i}\frac{\partial w}{\partial x_i}.$$

Furthermore:

$\partial_{a_i}(z/w) = [w.\partial z/\partial x_i - z.\partial w/\partial x_i - g_i.z.w]/w^2.f_i =$
$= [w.(\partial z/\partial x_i - g_i z)/f_i] - [z.(\partial w/\partial x_i - g_i w)/f_i] - g_i.z.w/f_i]/w^2 =$
$= [w.\partial_{a_i}(z) - z.\partial_{a_i}(w)]/w^2 - (z.g_i/(w.f_i)).$

4. The aleph partial derivative of the power of a function.

Applying the result on the section 2 two, three or more times, we get, for z $\in$ F:

$\partial_{a_i}(z^2) = 2.z.\partial_{a_i}(z) + z^2.g_i/f_i,$
$\partial_{a_i}(z^3) = z.\partial_{a_i}(z^2) + z^2.\partial_{a_i}(z) + z^3.g_i/f_i = 3.z^2.\partial_{a_i}(z) + 2.z^3.g_i/f_i$

and from an inductive argument it can be proved that:

$$\partial_{a_i}(z^n) = n.z^{n-1}.\partial_{a_i}(z) + (n-1).z^n.g_i/f_i, \ \forall \ n \in \mathbf{N}.$$

5. Let us define the operators:

$$\partial_{a_i} \equiv [f_i, \partial G/\partial x_i], \ i = 1,2, \ldots ,n$$

where G $\in$ C^2(D) and $f_i \in C^1(D)$, $\forall$i = 1,2, ... ,n.

We can work with two operators of the previous type as follows:

$$[\partial_{a_i} + \partial'_{a_i}](z) = \partial_{a_i}(z) + \partial'_{a_i}(z).$$

Thus, if we have:

$$\partial_{a_i} \equiv [f_i, \partial G/\partial x_i], \ \partial'_{a_i} \equiv [f'_i, \partial G/\partial x_i]$$

it holds:

$$\partial_{a_i}(z) + \partial'_{a_i}(z) = [\partial z/\partial x_i - \partial G/\partial x_i.z]/f_i + [\partial z/\partial x_i - \partial G/\partial x_i.z]/f'_i =$$
$$= \frac{f_i + f'_i}{f_i.f'_i}.[\partial z/\partial x_i - z.\partial G/\partial x_i],$$

i.e. $\partial_{a_i} + \partial'_{a_i}$ defines a new operator of the same type ∂''_{a_i}, which depends of the generatig functions:

$$f''_i \equiv f_i.f'_i/[f_i + f'_i] \text{ and } g''_i \equiv \partial G/\partial x_i.$$

Moreover, it can be proven without difficulty that the addition defines a structure of abelian group into the class of the aleph partial operators of the form $[f_i, \partial G/\partial x_i]$.

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