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Homomorphisms similar to completely contractive homomorphims

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Abstract

Paulsen has proved that a unital homomorphism from an operator algebra contained in a C^*-algebra is similar to a completely contractive homomorphism if and only if it is completely bounded. In the present note we obtain a different characterization when the operator algebra is separable.

Key words: Homomorphism, similarity, completely contractive.

Homomorfismos similares a homomorfismos completamente contractivos

Resumen

Paulsen ha probado que un homomorfismo unital de un álgebra de operadores contenida en una álgebra-C* es similar a un homomorfismo completamente contractivo si y sólo si es completamente acotado. En la presente nota damos una caracterización distinta cuando el álgebra de operadores es separable.

Palabras claves: Homomorfismo, similaridad, completamente contractivo.

Introduction

Let $\mathcal{B}$ be a C^*-algebra, $\mathcal{A} \subset \mathcal{B}$ an operator algebra and $\boldsymbol{H}$ a Hilbert space. In [1] it is proved that every unital homomorphism $\rho: \mathcal{A} \to \boldsymbol{L(H)}$ is similar to a completely contractive homomorphism if, and only if, ρ is completely bounded. Moreover, when ρ is completely bounded it is well known that there exist a Hilbert space $\mathbf{K}$, operators $\mathbf{A},\mathbf{B} \in \boldsymbol{L(H,K)}$ and a representation $\pi: \mathcal{B} \to \boldsymbol{L(K)}$ such that

$$\rho(a) = A^*\pi(a)B, \quad \forall\, a \in A \tag{1}$$

The purpose of the present note is to prove the following generalization of the above result, and give an answer to the problem of similarity for completely contractive homomorphisms, when $\mathcal{A}$ is separable.

Theorem 1.- *Let $\boldsymbol{B}$ be a C^*-algebra, $\mathcal{A} \subset \mathcal{B}$ a separable operator algebra and $\boldsymbol{H}$ a Hilbert space. $\rho: \mathcal{A} \to \boldsymbol{L(H)}$ a unital homomorphism. Then ρ is similar to a completely contractive homomorphism if, and only if, there exist a Hilbert space $\mathbf{K}$, operators $\mathbf{A},\mathbf{B} \in \boldsymbol{L(H,K)}$ and one representation $\pi: \mathcal{B} \to \boldsymbol{L(K)}$ such that*

$$\inf\left\{\sum_{i=1}^{\infty}\left\|\rho(a_i) - \mathbf{A}^*\pi(a_i)\mathbf{B}\right\|^2\right\} < \infty \tag{2}$$

where the infimum is taken over all countable families linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$.

Let $\boldsymbol{H}$ be a Hilbert space, $\boldsymbol{L(H)}$ the algebra of all bounded operators over $\boldsymbol{H}$, $\mathcal{B}$ a C*-algebra with unit and $\mathcal{A}$ a subalgebra of $\mathcal{B}$ that contains the unit of $\mathcal{B}$. Such subalgebras are called **Operator Algebras**. An operator algebra is **separa-**

ble if it possesses a countable family of linear generators. Thus, the closure of the spanned subspace by the family is the whole subalgebra.

In the sequel, M_n will denote the n×n matrix over **C** and we set $M_n(\mathcal{A}) = M_n \otimes \mathcal{A}$ ($M_n(\mathcal{A})$ can be thought of as subspace of the C*-algebra $M(\mathcal{B})$). We will denote by $\boldsymbol{H}^n$ the direct sum of n copies of $\boldsymbol{H}$, with $n \in N$. If $\|\cdot\|$ is the norm of $\boldsymbol{H}$, then the norm $\|\cdot\|_n$ of the Hilbert space $\boldsymbol{H}^n$ is given by

$$\|\tilde{h}\|_n^2 = \|h_1\|^2 + \|h_2\|^2 + \ldots + \|h_n\|^2, \tag{3}$$

where $\tilde{h} = (h_1, h_2, \ldots, h_n)$

Given a linear map, $\rho : \mathcal{A} \to \boldsymbol{L(H)}$ for every $n \in \mathbf{N}$ the mapping $\rho_n : M_n(\mathcal{A}) \to \boldsymbol{L(H^n)}$ is defined as follows:

$$\rho_n((\mathbf{a}_{ij})) = (\rho(\mathbf{a}_{ij})), \text{ for } (\mathbf{a}_{ij}) \in M_n(\mathcal{A}). \tag{4}$$

It is known that the sequence $\{\|\rho_n\|_n : n \in N\}$ is increasing. The map ρ is called **completely bounded** if

$$\sup\{\|\rho_n\|_n : n \in \mathbf{N}\} < \infty; \tag{5}$$

in that case, we will write $\|\rho\|_{cb}$ to denote this supremum. Then $\|\cdot\|_{cb}$ is a norm on the space of all completely bounded maps. If $\|\rho\|_{cb} \leq 1$, then we say that ρ is **completely contractive**.

To prove our theorem we give the following lemmas. The norm given in lemma 3 is a modification of that given in the proof of theorem 8.1 of [3].

Let us recursively define the matrices $R_m \in M_n(\mathbf{C})$ with $n = 2^m$ by $R_0 = (1)$ and

$$R_{m+1} = \begin{pmatrix} R_m & R_m \\ R_m & -R_m \end{pmatrix}, \tag{6}$$

for m=0,1,2,...

Lemma 1.- (i) R_m *is invertible,* $R_m^{-1} = 2^{-m} R_m$ and therefore $\|R_m\|_n^2 = 2^m$

(ii) *If* $\boldsymbol{H}$ *is a Hilbert space and* $\tilde{h} = (h_1, h_2, \ldots, h_n) \in \boldsymbol{H}^n$ *and* $\tilde{k} = (k_1, k_2, \ldots, k_n) = R_m(\tilde{h})$, *then*

$$\sum_{i=1}^{n} \|k_i\|^2 = 2^m \sum_{i=1}^{n} \|h_i\|^2 \tag{7}$$

Proof: (i) Obvious, by using inductivety.

(ii) If, $\tilde{k} = R_m(\tilde{h})$, then

$$\sum_{i=1}^{n} \|k_i\|^2 = \|\tilde{k}\|_n^2 = \|R_m(\tilde{h})\|_n^2 \leq \\ \|R_m\|_n^2 \|\tilde{h}\|_n^2 = 2^m \sum_{t=1}^{n} \|h_t\| \tag{8}$$

Now, $2^m \tilde{h} = R_m(\tilde{k})$, therefore,

$$2^{2m} \|\tilde{h}\|_n^2 = \|2^m \tilde{h}\|_n^2 = \|R_m(\tilde{k})\|_n^2 \leq \\ \|R_m\|_n^2 \|\tilde{k}\|_n^2 = 2^m \|\tilde{k}\|_n^2 \tag{9}$$

So, dividing by 2^m we conclude that

$$\sum_{i=1}^{n} \|k_i\|^2 \geq 2^m \sum_{i=1}^{n} \|h_i\|^2 \tag{10}$$

By (8) and (10) the equality holds.■

Lemma 2.-*Under the hypothesis of Theorem 1, let* $\alpha \in R$, $\alpha \geq 1$ *and set* $|\cdot|$ *defined by*

$$|h|^2 = \inf\left\{\left\|\sum_{i=1}^{\infty} \pi(\mathbf{a}_i) B h_i\right\|^2 + \alpha \sum_{i=1}^{\infty} \|h_i\|^2 : \sum_{i=1}^{\infty} \rho(\mathbf{a}_i) h_i = h\right\} \tag{11}$$

where the infimum is taken over all countable families of linear generators $\{\mathbf{a}_i\}$ *of* $\mathcal{A}$ *and over all sequences* $\{h_i\}$ *of finite support (this means that only a finite number of elements of the sequence is non null). Then* $(\boldsymbol{H}, |\cdot|)$ *is a Hilbert space and* $|\cdot|$ *is equivalent to* $\|\cdot\|$

Proof: Clearly $|zh|^2 = |z|^2 |h|^2$, for all $z \in \mathbf{C}$. Thus,

$$|zh| = |z|\,|h|, \tag{12}$$

If $h = \sum_{i=1}^{\infty} \rho(a_i) h_i$ and $k = \sum_{i=1}^{\infty} \rho(a_i) k_i$ where $\{\mathbf{a}_i\}$ is any countable family of linear generators of $\mathcal{A}$, then $h + k = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)(h_i + k_i)$. Therefore

$$|h+k| \leq \inf\left\{\left\|\sum_{i=1}^{\infty} \pi(\mathbf{a}_i) \mathbf{B}(h_i + k_i)\right\|^2 + \alpha \sum_{i=1}^{\infty} \|h_i + k_i\|^2\right\}^{1/2}$$

$$= \inf\left\{\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i + \sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}k_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\left\|h_i + k_i\right\|^2\right\}^{1/2}$$

$$= \inf\left\{\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i + \sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}k_i\right\|^2 + \sum_{i=1}^{\infty}\left\|\alpha^{1/2}(h_i + k_i)\right\|^2\right\}^{1/2}$$

$$\leq \inf\left\{\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\left\|h_i\right\|^2\right\}^{1/2} + \inf\left\{\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}k_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\left\|k_i\right\|^2\right\}^{1/2}$$

$$= |h| + |k|$$

where the last inequality is due to the triangular inequality for l_2. So, we get

$$|h + k| \leq |h| + |k| \qquad (13)$$

By (12) and (13), it follows that $|\cdot|$ is a seminorm in $\boldsymbol{H}$.

As ρ is unital, it yields that

$h = \rho(I)h + \sum_{i=1}^{\infty}\rho(a_i)0$, where $\{\mathbf{a}_i\}$ is any countable family of linear generators of $\mathcal{A}$.

So,

$$|h|^2 \leq \|\rho(I)\mathbf{B}h\|^2 + \|h\|^2 = \|\mathbf{B}h\|^2 + \|h\|^2$$

Thus,

$$|h| \leq (\|\mathbf{B}h\|^2 + \|h\|^2)^{1/2} = (\|\mathbf{B}\|^2\|h\|^2 + \|h\|^2)^{1/2} = (\|\mathbf{B}\|^2 + 1)^{1/2}\|h\|$$

$$|h| \leq (\|\mathbf{B}\|^2 + 1)^{1/2}\|h\| \qquad (14)$$

On the other hand, if $h = \sum_{i=1}^{\infty}\rho(\mathbf{a}_i)h_i$, then

$$\|h\| = \left\|\sum_{i=1}^{\infty}\rho(a_i)h_i\right\| = \left\|\mathbf{A}^*\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i + \sum_{i=1}^{\infty}(\rho(a_i) - \mathbf{A}^*\pi(a_i)\mathbf{B})h_i\right\|$$

$$\leq \left\|\mathbf{A}^*\right\|\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i\right\| + \sum_{i=1}^{\infty}\left\|\rho(a_i) - \mathbf{A}^*\pi(a_i)\mathbf{B}\right\|\left\|h_i\right\|$$

$$\leq \left[\left(\left\|\mathbf{A}^*\right\|^2 + \sum_{i=1}^{\infty}\left\|\rho(a_i) - \mathbf{A}^*\pi(a_i)\mathbf{B}\right\|^2\right)\right]\left[\left(\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i\right\|^2 + \sum_{i=1}^{\infty}\left\|h_i\right\|^2\right)\right]^{1/2} \quad \left(\text{by Schwarz inequality for } l_2\right)$$

$$\leq \left[\left(\left\|\mathbf{A}^*\right\|^2 + \sum_{i=1}^{\infty}\left\|\rho(a_i) - \mathbf{A}^*\pi(a_i)\mathbf{B}\right\|^2\right)\right]\left[\left(\left\|\sum_{i=1}^{\infty}\pi(a_i)\mathbf{B}h_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\left\|h_i\right\|^2\right)\right]^{1/2}$$

since $\alpha \geq 1$. Taking infimum over all sequences $\{h_i\}$ of finite support and all countable families of linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$ such that, $h = \sum_{i=1}^{\infty}\rho(\mathbf{a}_i)h_i$ we gets by (2) and the definition $|h|$ that

$$\|h\| \leq k|h|, \qquad (15)$$

where $k = \left(\left\|\mathbf{A}^*\right\|^2 + \inf\left\{\sum_{i=1}^{\infty}\left\|\rho(\mathbf{a}_i) - \mathbf{A}^*\pi(\mathbf{a}_i)\mathbf{B}\right\|^2\right\}\right)^{1/2}$

where the infimum is taken over all countable families of numbering linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$.

By (12), (13) and (15) we conclude that $|\cdot|$ is a norm on $\boldsymbol{H}$. Then (14) and (15) show that $|\cdot|$ is equivalent to $\|\cdot\|$. As in [2] it suffices to show that $|\cdot|$ satisfies the parallelogram law, to prove that the pair $(\boldsymbol{H}, |\cdot|)$ is a Hilbert space. Indeed,

Let $h = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)h_i$ and $h' = \sum_{i=1}^{\infty} \rho(\mathbf{b}_i)h'_i$,

where $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$ are any countable families of linear generators of $\mathcal{A}$.

Then

$$h + h' = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)h_i + \sum_{i=1}^{\infty} \rho(\mathbf{b}_i)h'_i \tag{16}$$

and

$$h - h' = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)h_i - \sum_{i=1}^{\infty} \rho(\mathbf{b}_i)h'_i \tag{17}$$

Now, we have

$$\|h+h'\|^2 + \|h-h'\|^2 \le$$

$$\left\|\sum_{i=1}^{\infty}\pi(\mathbf{a}_i)\mathbf{B}h_i + \sum_{i=1}^{\infty}\pi(\mathbf{b}_i)\mathbf{B}h'_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\|h_i\|^2 - \alpha\sum_{i=1}^{\infty}\|h'_i\|^2 +$$

$$\left\|\sum_{i=1}^{\infty}\pi(\mathbf{a}_i)\mathbf{B}h_i - \sum_{i=1}^{\infty}\pi(\mathbf{b}_i)\mathbf{B}h'_i\right\|^2 + \alpha\sum_{i=1}^{\infty}\|h_i\|^2 - \alpha\sum_{i=1}^{\infty}\|h'_i\|^2$$

$$= 2\left(\left\|\sum_{i=1}^{\infty}\pi(\mathbf{a}_i)\mathbf{B}h_i\right\|^2 + \left\|\sum_{i=1}^{\infty}\pi(\mathbf{b}_i)\mathbf{B}h'_i\right\|^2 + \right)$$

$$\left(\alpha\sum_{i=1}^{\infty}\|h_i\|^2 - \alpha\sum_{i=1}^{\infty}\|h'_i\|^2\right)$$

In the last inequality we used the parallelogram law in $\boldsymbol{K}$. Taking infimum first over the sequences $\{h_i\}$ and all countable families of linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$, for which, $h = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)h_i$, and, then over the sequences $\{h'_i\}$ and all countable families of linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$, for which, $h' = \sum_{i=1}^{\infty} \rho(\mathbf{b}_i)h'_i$, we get

$$|h+h'|^2 + |h-h'|^2 \le 2\left(|h|^2 + |h'|^2\right) \tag{18}$$

Let us note that replacing h by $h+h'$ and h' by $h-h'$ in (18) one gets the reciprocal inequality.■

Proof of Theorem 1

If $\rho:\mathcal{A} \to L(H)$ is similar to a completely contractive homomorphism $\varphi:\mathcal{A} \to L(H)$, then by Corollary 6.7 in [3], there exists a representation $\varphi:\mathcal{B} \to L(\mathbf{K})$, where $\boldsymbol{K}$ is some Hilbert space that contains $\boldsymbol{H}$, such that, $\varphi(\mathbf{a}) = \mathbf{P}\pi(\mathbf{a})\boldsymbol{i}$, for all $\mathbf{a} \in \mathcal{A}$, where $\mathbf{P}$ denotes the orthogonal projection of $\boldsymbol{K}$ onto $\boldsymbol{H}$ and $\boldsymbol{i}$ is the inclusion from $\boldsymbol{H}$ to $\boldsymbol{K}$. As $\rho(\mathbf{a}) = \mathbf{S}^{-1}\varphi(\mathbf{a})\mathbf{S}$, for all $\mathbf{a} \in \mathcal{A}$ and for some invertible operator $\mathbf{S} \in L(H)$, it yields,

$$\rho(\mathbf{a}) = \mathbf{S}^{-1}\mathbf{P}\pi(\mathbf{a})\boldsymbol{i}\mathbf{S},\ \forall\ \mathbf{a} \in \mathcal{A} \tag{19}$$

Taking $\mathbf{A} = (\mathbf{S}^{-1}\mathbf{P})^*$, $\mathbf{B} = \boldsymbol{i}\mathbf{S}$, the representation π and the space $\boldsymbol{K}$, we obtain for all countable families of linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$ that,

$$\sum_{i=1}^{\infty} \left\|\rho(\mathbf{a}_i) - \mathbf{A}^*\pi(\mathbf{a}_i)\mathbf{B}\right\|^2 = 0 \tag{20}$$

Therefore, the infimum over such families is finite.

Conversely, let $|\cdot|$ be the norm of $\boldsymbol{H}$ as defined in Lemma 2, with $\alpha = 1$. Wee will show that with respect to such a norm ρ is a completely contractive map. Indeed, if $h = \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)h_i$, and $\mathbf{a} \in \mathcal{A}$ with $\|\mathbf{a}\| = 1$, then

$$\rho(\mathbf{a})h = \sum_{i=1}^{\infty} \rho(\mathbf{a}\mathbf{a}_i)h_i \tag{21}$$

If $\{\mathbf{a}\mathbf{a}_i\}$ is not a countable family of linear generators of $\mathcal{A}$. Then this can always be extended to a countable family of linear generators of $\mathcal{A}$, since, for $\{\mathbf{a}\mathbf{a}_i\} \cup \{\mathbf{a}_i\}$, we have

$$\rho(\mathbf{a})h = \sum_{i=1}^{\infty} \rho(\mathbf{a}\mathbf{a}_i)h_i + \sum_{i=1}^{\infty} \rho(\mathbf{a}_i)0 \tag{22}$$

Therefore,

$$|\rho(\mathbf{a})h|^2 \le \left\|\sum_{i=1}^{\infty} \pi(\mathbf{a}\mathbf{a}_i)\mathbf{B}h_i\right\|^2 + \sum_{i=1}^{\infty}\|h_i\|^2$$

$$= \left\| \pi(\mathbf{a}) \sum_{i=1}^{\infty} \pi(\mathbf{a}_i)\mathbf{B}h_i \right\|^2 + \sum_{i=1}^{\infty} \|h_i\|^2$$

$$\leq \left\| \sum_{i=1}^{\infty} \pi(\mathbf{a}_i)\mathbf{B}h_i \right\|^2 + \sum_{i=1}^{\infty} \|h_i\|^2 \qquad (23)$$

where the last inequality is due to the fact that $\pi(\mathbf{a})$ is a contraction. Taking the infimum over the sequences $\{h_i\}$ and the countable families of linear generators $\{\mathbf{a}_i\}$ of $\mathcal{A}$, such that, one gets

$$|\rho(\mathbf{a})h|^2 \leq |h|^2 \qquad (24)$$

Thus $\rho(\mathbf{a})$ is a contraction in $L(H)$ with respect to the norm $|\cdot|$ on H. Therefore ρ is contractive.

Now, we must prove that $|\rho_n|_n \leq 1$ for all $n \in \mathbb{N}$. Since the sequence $(|\rho_n|_n)$ is increasing on H it suffices to prove that the inequality holds for all $n=2^m$ with $m \in \mathbb{N}$.

First, note that if $\{\mathbf{a}_i\}$ is a countable family of linear generators of $\mathcal{A}$, such that, $\sum_{i=1}^{\infty} \|\rho(\mathbf{a}_i) - \mathbf{A}^*\pi(\mathbf{a}_i)\mathbf{B}\|^2 < \infty$, then the family $\{(\mathbf{a}_i\mathbf{E}_{jk}): i=1,2,\ldots, \text{ and } j,k = 1,2,\ldots,n\}$, (where $\mathbf{E}_{jk}$ is the $n\times n$ matrix with the unit of $\mathcal{A}$ in the (j,k)-entry and zero elsewhere) generates $\mathbf{M}_n(\mathcal{A})$. Moreover,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{\infty} \left\|\rho(\mathbf{a}_i(\mathbf{E}_{jk})) - \tilde{\mathbf{A}}^*\pi(\mathbf{a}_i(\mathbf{E}_{jk}))\tilde{\mathbf{B}}\right\|^2 < \infty \qquad (25)$$

where $\tilde{A} = A\otimes I$ and $= B\otimes I$.

We define the following norm $|\cdot|_1$, for all $h \in H$,

$$|\tilde{h}|_1^2 = \inf\left\{ \left\| \sum_{i=1}^{\infty} \pi_n(\tilde{\mathbf{a}}_i)\tilde{\mathbf{B}}\tilde{h}_i \right\|_n^2 + n\sum_{i=1}^{\infty}\|\tilde{h}_i\|_n^2 : \sum_{i=1}^{\infty}\rho_n(\tilde{\mathbf{a}}_i)\tilde{h}_i = \tilde{h} \right\} \qquad (26)$$

where the infimum is taken as in Lemma 2, over all countable families of linear generators $\{\tilde{a}_i\}$ of $M_n(\mathcal{A})$, etc. By (26) and Lemma 2 $|\cdot|_1$ makes H^n into a Hilbert space. Moreover $|\cdot|_1$ is equivalent to $\|\cdot\|_n$, and by the argument used in the case of H, it follows that ρ_n is contractive with respect to this norm. To conclude, we will prove that

$$|\tilde{h}|_n = |\tilde{h}|_1 \quad \forall\ \tilde{h} \in H^n \quad \text{(see(3))}$$

Let $\tilde{h} = (h_1, h_2, \ldots, h_n)$ and fix $\varepsilon > 0$. Let $\tilde{\mathbf{a}}_k = (\mathbf{a}_{ijk})$ and $\tilde{h}_k = (h_{1k}, h_{2k}, \ldots, h_{nk})$, be such that, $\tilde{h} = \sum_{k=1}^{\infty} \rho(\tilde{\mathbf{a}}_k)\tilde{h}_k$ and

$$|\tilde{h}|_1^2 + \varepsilon \geq \left\| \sum_{k=1}^{\infty} \pi_n(\tilde{\mathbf{a}}_k)\tilde{\mathbf{B}}\tilde{h}_k \right\|_n^2 + n\sum_{i=1}^{\infty}\|\tilde{h}_k\|_n^2 \qquad (27)$$

As $h_i = \sum_{k=1}^{\infty}\sum_{j=1}^{n} \rho(\mathbf{a}_{ijk})h_{jk}$, one has

$$|\tilde{h}|_n^2 = \sum_{i=1}^{n}|h_i|^2 \leq$$

$$\sum_{i=1}^{n}\left(\left\| \sum_{k=1}^{\infty}\sum_{j=1}^{n} \pi(a_{ijk})B_{jk} \right\|^2 + \sum_{k=1}^{\infty}\sum_{j=1}^{n} \|h_{jk}\|^2 \right)$$

$$= \left\| \sum_{k=1}^{\infty} \pi_n(\tilde{a}_k)\tilde{B}\tilde{h}_k \right\|_n^2 + n\sum_{i=1}^{\infty}\|\tilde{h}_k\|_n^2 \leq |\tilde{h}|_1^2 + \varepsilon$$

Therefore,

$$|\tilde{h}|_n^2 \leq |\tilde{h}|_1^2 \qquad (28)$$

On the other hand, let $\tilde{h} = (h_1, \ldots, h_n)$ and

$$h_i = \sum_{l=1}^{\infty} \rho(\mathbf{a}_{il})h_{il}, \qquad (29)$$

be such that, $|h_i|^2 + \frac{\varepsilon}{n} \geq \left\| \sum_{l=1}^{\infty} \pi(\mathbf{a}_{il})\mathbf{B}h_{il} \right\|^2 + \sum_{l=1}^{\infty}\|h_{il}\|^2$.

As $n = 2^k$, let us consider the matrix $\tilde{c}_l$, given below:

$$\begin{pmatrix} r_{11}\mathbf{a}_{1l} & r_{11}\mathbf{a}_{1l} & \cdots & r_{1n}\mathbf{a}_{1l} \\ r_{21}\mathbf{a}_{2l} & r_{22}\mathbf{a}_{1l} & \cdots & r_{2n}\mathbf{a}_{2l} \\ \vdots & \vdots & & \vdots \\ r_{n1}\mathbf{a}_{nl} & r_{n2}\mathbf{a}_{nl} & \cdots & r_{nn}\mathbf{a}_{nl} \end{pmatrix} \qquad (30)$$

where $(r_{ij}) = R_k$ is the matrix given in Lemma 1. Let $\tilde{h}_l = (h_{1l}, h_{2l}, \ldots, h_{nl})$ and $\tilde{k}_l = (k_{1l}, k_{2l}, \ldots, k_{nl}) = R_m(\tilde{h}_l)$

Let $\beta = n^{-1}$; by Lemma 2 one has that $\tilde{h}_l = R_m(\beta\tilde{k}_l)$. This means that

$$h_{il} = \sum_{j=1}^{n} r_{ij}\beta k_{jl}, \tag{31}$$

and

$$h_i = \sum_{l=1}^{\infty} \rho(\mathbf{a}_{il})\sum_{j=1}^{n} r_{ij}\beta k_{jl} = \sum_{l=1}^{\infty}\sum_{j=1}^{n} \rho(r_{ij}\mathbf{a}_{il})\beta k_{jl} \tag{32}$$

Therefore,

$$\tilde{h} = \sum_{l=1}^{\infty} \rho_n(\tilde{\mathbf{c}}_l)\tilde{\mathbf{B}}\beta\tilde{k}_l \tag{33}$$

Moreover,

$$\begin{aligned} \|\tilde{h}\|_l^2 &\leq \left\|\sum_{l=1}^{\infty} \pi_n(\tilde{\mathbf{c}}_l)\tilde{\mathbf{B}}\beta\tilde{k}_l\right\|_n^2 + n\sum_{l=1}^{\infty}\|\tilde{k}_l\|_n^2 = \\ &= \sum_{i=1}^{n}\left\|\sum_{l=1}^{\infty}\sum_{j=1}^{n}\pi(r_{ij}\mathbf{a}_{il})\mathbf{B}\beta k_{jl}\right\|^2 + \sum_{l=1}^{\infty}\beta\sum_{j=1}^{n}\|k_{jl}\|^2 = \\ &= \sum_{i=1}^{n}\left\|\sum_{l=1}^{\infty}\sum_{j=1}^{n}\pi(\mathbf{a}_{il})\mathbf{B}h_{jl}\right\|^2 + \sum_{l=1}^{\infty}\sum_{j=1}^{n}\|h_{jl}\|^2 \leq \|\tilde{h}\|_n^2 + \varepsilon, \end{aligned} \tag{34}$$

where we made use of (7) and the fact that $\beta = 2^{-k}$

Thus, the reverse inequality is proved and we conclude the proof of the theorem.■

As an application of the above result to the Operator Theory, we will characterize those operators that are similar to contractions. It is known that $\mathbf{T} \in L(H)$ is similar to a contraction if, and only if, the homomorphism ρ:P(**D**) $\to L(H)$ to defined by

$$\rho(f) = f(\mathbf{T}) \tag{35}$$

is completely bounded, where P(**D**) denotes the space of the polynomials on the unit disc(see theorem 8.11 of [3]).Then by the result of Paulsen mentioned in the introduction, it suffices to prove that ρ is completely contractive homomorphism. But, in virtue of the Theorem 1, it reduces to find a Hilbert space $\mathbf{K}$, a contraction $\mathbf{C} \in L(K)$, operators $\mathbf{A},\mathbf{B} \in L(H,K)$, and a countable family $\{f_i\}$ of linear generators of P(**D**) such that

$$\sum_{i=1}^{\infty} \|f_i(\mathbf{T}) - \mathbf{A}^* f_i(\mathbf{C})\mathbf{B}\|^2 < \infty. \tag{36}$$

since the homomorphism π:C(**T**) $\to L(K)$ defined by π:(f = f(**C**) is completely bounded. In particular, if the family of generators is $f_i = z^i$, with $i = 0,1,2,\ldots$, we obtains the following corollary.

Corollary 1.-(Holbrook Theorem [2]) *Suppose $\mathbf{T} \in L(H)$. Then $\mathbf{T}$ is similar to a contraction if, and only if, there exist a Hilbert space $\mathbf{K}$, a contraction $\mathbf{C} \in L(K)$ and operators $\mathbf{A},\mathbf{B} \in L(H,K)$, such that,*

$$\sum_{i=0}^{\infty} \|\mathbf{T}^i - \mathbf{A}^*\mathbf{C}^i\mathbf{B}\|^2 < \infty. \tag{37}$$

From this corollary, one can obtain the results of Rota [4]and the Sz-Nagy and Foias [5] about similarity to contractions by more direct procedures than those used in the proof by Paulsen in [1]. See Holbrook [2].

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