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Isotopic lifting of analytic and quantum mechanics

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Abstract

In a preceding article we have introduced the isotopies of the differential calculus and of Newton's equations of motion. In this second paper we use these results to construct the isotopies of analytic and quantum mechanics. We show that the isotopies of Hamiltonian mechanics permit the derivation from a first-order isovariational principle of the most general possiblenonlinear integro-differential Newton's equations by providing in particular a representation of the extended and deformable shape of the body considered as well as of nonlocal-integral and variationally non-self-adjoint forces. We then identify the isotopies of conventional quantization and show that they lead to unique and unambiguous isotopies of quantum mechanics capable of preserving all the essential characteristics of the original isotopic Newton's equations, thus permitting the representation in the fixed inertial frame of the experimenter of nonlinear, nonlocal and nonhamiltonian systems, with considerable broadening of the arena of applicability of conventional formulations.

Key words: Isotopies, isolagrangian and isohamiltonian mechanics, hadronic mechanics.

Levantamiento isotópico de la mecánica analítica y cuántica

Resumen

En un artículo anterior introdujimos las isotopías del cálculo diferencial y de las ecuaciones newtonianas de movimiento. En este segundo trabajo utilizamos estos resultados para construir las isotopías de la mecánica cuántica y analítica. Demostramos que las isotopías de la mecánica hamiltoniana permiten la derivación de un principio isovariacional de primer orden de las ecuaciones entero-diferenciales no lineales más generales posibles de Newton proporcionando, en particular, una representación de la forma extendida y deformada del cuerpo considerado, al igual que de fuerzas no locales-integrales y variacionalmente no auto-adjuntas. Luego identificamos las isotopías de la cuantificación convencional y demostramos que llevan a isotopías únicas y no ambigüas de la mecánica cuántica que son capaces de preservar todas las características esenciales de las ecuaciones isotópicas originales de Newton, permitiendo así la representación en el marco fijo inerte del experimentador de sistemas no lineales, no locales y no hamiltonianos, con una ampliación considerable del campo de aplicabilidad de formulaciones convencionales.

Palabras claves: Isotopías, mecánica isolagrangiana e isohamiltoniana, mecánica hadrónica.

1. Statement of the problem.

In the preceding paper [21] (hereinafter referred as Paper I) we have: reviewed the main elements of nonlinear, nonlocal and nonhamiltonian, yet axiom-preserving maps called *isotopies*; introduced the *isodifferential calculus*; and constructed the *isotopic Newton's equations of motion*. We have then shown that these new methods permit the representation of the extended and deformable shapes of the particles considered as well as of their nonlocal-integral and *variationally non-self-adjoint* interactions (NSA) [7,14] (i.e., interactions violating the integrability conditions for the existence of a potential).

In this paper we study the isotopies of conventional classical and quantum mechanics. Their primary motivation is the following. Conventional analytic mechanics is derivable from a first-order action principle either in the familiar Lagrangian form on $S(t, x, v) = E(t)\times E(x,\delta,R)\times E(v,\delta,R)$, where $E(v,\delta,R)$ is the tangent space of $E(x,\delta,R)$, or in the equivalent canonical form on $S(t,x,p) = E(t)\times E(r,\delta,R)\times E(p,\delta,R)$, where $E(p,\delta,R)$ is the cotangent space to $E(x,\delta,R)$ (see Paper I for all notations)

$$\delta\int_{t_1}^{t_2} L(t, x, v)\, dt = \delta\int_{t_1}^{t_2}[\, p_k\, dx^k - H(t, x, p)\, dt\,] =$$

$$= \delta\int_{t_1}^{t_2}[\, R^\circ_\mu(b)\, db^\mu - H(t, b)\, dt\,] = 0, \tag{1.1a}$$

$$R^\circ = \{ R^\circ_\mu \} = \{ p_k, 0 \}, \quad k = 1, 2, ..., N, \quad \mu = 1, 2, ..., 2N. \tag{1.1b}$$

The contemporary formulation of Lagrange equations along an actual path P° are then given by

$$\left\{ \frac{d}{dt}\frac{\partial L(t, x, v)}{\partial v^k} - \frac{\partial L(t, x, v)}{\partial x^k} \right\}(P^\circ) = 0, \tag{1.2}$$

while the corresponding contemporary form of Hamilton's equations in the unified notation $b = \{b^\mu\} = \{r^k, p_k\}$ is

$$\left\{ \omega_{\mu\nu}\frac{db^\nu}{dt} - \frac{\partial H(t, b)}{\partial b^\mu} \right\}(P^\circ) = 0, \tag{1.3}$$

where $\omega_{\mu\nu}$ is the familiar *exact canonical symplectic tensor*

$$(\omega_{\mu\nu}) = (\partial_\mu R^\circ_\nu - \partial_\nu R^\circ_\mu) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix} \tag{1.4}$$

The fundamental problem addresses in this paper is that the above analytic equations can only represent a rather small class of Newtonian systems in the fixed local coordinates. In fact, the equations can only represent Newtonian systems which are local-differential and selfadjoint, such as our planetary systems. More general systems such as the equations of motion of a satellite during re-entry in our atmosphere (see Paper I) are outside the representational capabilities of the above equations.

When the restriction to the local chart $\{t,x,v\}$ or $\{t,x,p\}$ is removed and coordinate transformations are admitted, principle (1.1) can represent all possible analytic and regular non-self-adjoint Newtonian systems in a star-shaped region of the variables, provided that they are still *local-differential* (this is the *Lie-Koening theorem* [15] as the analytic counterpart of the *Darboux's theorem* of the symplectic geometry studied in the next paper).

Even though evidently correct on *mathematically* grounds, the latter representation has serious *physical* drawbacks which prevent its practical use. First, the transformations needed for the reduction of a nonhamiltonian system in the given frame to a Hamiltonian form in another frame are nonlinear and, as such, *the coordinates of the equivalent Hamiltonian form are not realizable in laboratory*. Also, their nonlinearity implies the *loss of the original inertial character of the reference frame with consequential loss of conventional relativities* (in fact, the Galilei and Einstein relativities are solely applicable to inertial systems, as well known, thus preventing the use of the Lie-Koening and Darboux Theorems).

The fundamental problem in analytic dynamics addressed in this paper is therefore *the construction of an analytic representation of the most general possible, nonlocal-integral and non-self-adjoint Newtonian systems in the fixed local chart x representing the inertial system of the experimenter*. After the achievement of this representation, then the use of the transformation theory may have a physical relevance.

The first solution of the above problem was reached by the originators of analytic dynamics, Lagrange and Hamilton themselves, because they formulated their celebrated equations, not in the form of current use in the mathematics and physics, Eq.s (1.2) and (1.3), but that with *external terms*. The "true" Lagrange's equations are then given by [8]

$$\left\{\frac{d}{dt}\frac{\partial L(t,x,v)}{\partial v^k} - \frac{\partial L(t,x,v)}{\partial x^k} - F^{NSA}{}_k(t,x,v)\right\}(P^\circ) = 0 \tag{1.5}$$

while the "true" Hamilton 's equations can be written in unified notation [6]

$$\left\{\omega_{\mu\nu}\frac{db^\nu}{dt} - \frac{\partial H(t,b)}{\partial b^\mu} - F_\mu{}^{NSA}(t,b)\right\}(P^\circ) = 0,$$

$$\{F_\mu\} = \{F_k, 0\}. \tag{1.6}$$

According to the above historical conception, the functions today called Lagrangian or Hamiltonian represent all potential forces, while all remaining forces are represented with the external terms. The above representation of Newton's equations does indeed verify the crucial requirement of occurring in the fixed inertial x-frame of the experimenter, and the construction of the representation is simple and immediate. Unfortunately, analytic equations (1.5) and (1.6) are not generally derivable from a variational principle, thus preventing the use of all related methods, such as the optimal control theory. Moreover, the brackets among two functions A(b), B(b) on the cotangent bundle characterized by Hamilton's equations with external terms,

$$(A,B) = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu} + F_\mu\frac{\partial B}{\partial b^\mu}, \quad \omega^{\mu\nu} = [(\omega_{\alpha\beta})^{-1}]^{\mu\nu}, \tag{1.7}$$

violate the left scalar and distributive laws and, as such, they do not characterize any algebra as conventionally understood. This implies the inapplicability of all methods of contemporary physics based on Lie's theory.

In order to resolve the latter problem, Santilli [12,15] reformulated brackets (1.7) in the form

$$(A,B) = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu} + F_\mu\frac{\partial B}{\partial b^\mu} = \frac{\partial A}{\partial b^\mu}S^{\mu\nu}\frac{\partial B}{\partial b^\nu}, \tag{1.8a}$$

$$S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu}, \quad s = (s^{\mu\nu}) =$$

$$= \mathrm{diag.}\,(I_{N\times N}, F_{k\partial}/\,\partial H/\partial p_k), \tag{1.8b}$$

which now verify the left and right scalar and distributive laws, thus characterizing an algebra. However, brackets (1.8) are not totally antisymmetric and they therefore violate the axioms of Lie algebras in favor of the more general *Lie-admissible algebras* identified by Albert [1], i.e., the antisymmetric brackets [A, B] = (A, B) − (B, A) verify the Lie algebra axioms. It then follows that the geometry underlying the latter brackets cannot be the symplectic geometry, because requiring a generalization of the calculus of differential forms which is no longer totally antisymmetric. We can therefore state that the analytic equations according to Hamilton's original conception are structurally beyond contemporary analytic, algebraic and geometric methods.

A solution of the fundamental analytic problem here considered which preserves the *Lie* character of the underlying algebra, with consequential preservation of its *symplectic* geometry, was reached by Santilli in monograph [15] via a step-by-step isotopic generalization of Hamiltonian mechanics called, for certain historical reasons, *Birkhoffian mechanics*. The main idea is to lift the canonical action principle (1.1) into the most general possible first-order action of the Pfaffian [11] type

$$\delta A = \int_{t_1}^{t_2}[R_\mu(b)\,db^\mu - H(t,b)\,dt] = 0,$$

$$R = \{P_k(x,p), Q^k(x,p)\}, \tag{1.9}$$

which characterizes Birkhoff's equations [3],

$$\left\{\Omega_{\mu\nu}(b)\frac{db^\nu}{dt} - \frac{\partial H(t,b)}{\partial b^\mu}\right\}(P^\circ) = 0, \tag{1.10}$$

where $\Omega_{\mu\nu}$ is an exact, nowhere degenerate and therefore symplectic tensor although in its most general possible realization (see next paper for geometric details)

$$\Omega_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu . \qquad (1.11)$$

A "Theorem of Direct Universality" was proved in ref. [15], Sect. 4.5, according to which Pfaffian actions (1.9) can represent in a star-shaped region D[S(t,x,p)] all possible analytic, regular and variationally non-self-adjoint Newtonian systems (universality) directly in the x-frame of the experimenter (direct universality).

Birkhoffian mechanics resulted to be a particular isotopy of Hamiltonian mechanics because preserving at the abstract level all original analytic, geometric and algebraic properties. Most importantly, the representation via Birkhoff's equations of non-self-adjoint systems preserves the Lie character of the underlying algebra and the symplectic character of the underlying geometry, although expressed in their most general possible regular form [12,15].

The above approach permitted the resolution of the major drawback in the use of the historical equations (1.6), the loss of Lie's theory. However, the approach has the limitation that *Birkhoffian mechanics can only represent local-differential systems*, due to the strictly local-differential character of the underlying symplectic geometry.

Upon achieving the above partial solution, this author (a particle physicist) was forced to seek a more adequate analytic representation of sufficiently smooth and regular, but otherwise arbitrary, linear and nonlinear, local and nonlocal, self-adjoint and non-self-adjoint systems (universality), in the fixed inertial frame of the experimenter (direct universality). Such a *classical* solution was necessary for the initiation of quantitative studies on the historical open legacy due to Bloch'intsev, Fermi and others that the strong interactions have a nonlocal-integral component due to mutual overlapping of the wavepackets and charge distributions of hadrons (in fact, all hadrons have approximately the same size which coincides with the range of the strong interactions, thus requiring the necessary condition of mutual penetration of hyperdense particles, resulting in the most general known nonlinear integro-differential equations).

In this paper we present, apparently for the first time, a solution of the fundamental problem herein considered along the latter lines, which is permitted by the isotopies of the differential calculus and of Newton's equations of the preceding Paper I. The solution is uniquely and unambiguously characterized by N-dimensional isounits of Kadeisvili topological class I (sufficiently smooth, bounded, nowhere singular, real valued, symmetric and positive-definite, see Paper I) with nonlinear and nonlocal-integral dependence on coordinates x, their derivatives $\dot{x}$, $\ddot{x}$, ..., with respect to an independent variable t and any additional variable needed in applications. In their diagonal form, the isounits can be written

$$\hat{1} = \mathrm{diag.}\,(n_1^{-2}, n_2^{-2}, n_3^{-2})\,\Gamma(x, \dot{x}, \ddot{x}, ...)\,, \qquad (1.12)$$

where diag. $(n_1^{-2}, n_2^{-2}, n_3^{-2})$ represents the shape of the particle considered and $\Gamma(x, \dot{x}, \ddot{x}, ...)$ represents its nonlinear, nonlocal and nonhamiltonian interactions. Conventional action-at-a-distance interactions are represented via the conventional potential.

We initiate our studies with the identification, also done here for the first time, of the conditions of self-adjointness in isospace, and then pass to the study of analytic mechanics on isospaces over isofields. The paper ends with the identification of a simple, yet unique and unambiguous isotopy of conventional quantization which implies an isotopic lifting of quantum mechanics capable of preserving all original characteristics of the isotopic Newton's equations, including nonlocal-integral forces, as desired for novel treatments of strong interactions.

Our analysis is strictly local, owing to the need to identify methods which are specifically applicable in the given inertial frame of the observer. All results of this paper can be easily extended to isounits of Kadeisvili Class II (same property of Class I except that $\hat{1}$ is negative-definite) and of Class III (union of Class I and II). However the extension to Classes IV (Class III plus singular isounits) and V (Class IV plus arbitrary isounits, including discontinuous isounits) requires specific studies.

2. Variational iso-self-adjointness.

The fundamental methods of the Inverse Newtonian Problem are the conditions of variational self-adjointness in $E(t)\times E(x,\delta,R)\times E(v,\delta,R)$ [7,14]. In this section we identify the corresponding *conditions of variational iso-self-adjointness* in isospaces over isofields.

Theorem 1

A necessary and sufficient condition for a system of ordinary second-order isodifferential equations in $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t})\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta},\hat{R})$

$$\hat{F}_k(\hat{t}, \hat{x}, \hat{v}, \hat{a}) = 0, \; k = 1, 2, ..., N, \; \hat{v} = \hat{\partial}\hat{x}/\hat{\partial}\hat{t}, \; \hat{a} = \hat{\partial}\hat{v} / \hat{\partial}\hat{t}, \tag{2.1}$$

which are isodifferentiable at least up to the third order and regular in a region $\hat{D}(\hat{t},\hat{x},\hat{v})$ *of points* $(\hat{t}, \hat{x}, \hat{v}, \hat{a}, \hat{\partial}\hat{a}/\hat{\partial}\hat{t})$ *(i.e.,* $\det \hat{\partial}\hat{F}_i / \hat{\partial}\hat{a}^j)(\hat{D}) \neq 0$*) to be variationally iso-self-adjoint* (ISOSA) *in* $\hat{D}$ *is that all the following conditions*

$$\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^k} = \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{a}^i}, \tag{2.2a}$$

$$\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{v}^k} + \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{v}^i} = 2\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^k} = \frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^k} + \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{a}^i}\right), \tag{2.2b}$$

$$\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{x}^k} - \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{x}^i} = \frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left[\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{v}^i}\right) - \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{v}^i}\right] = \frac{1}{2}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^k} - \frac{\hat{\partial}\hat{F}_k}{\hat{\partial}\hat{a}^i}\right), \tag{2.2c}$$

are identically verified in $\hat{D}$.

The proof is provided by an elementary isotopy of the conventional case, ref. [14], Theorem 2.1.2, p. 60, and consists in computing the isovariational forms of system (2.1), proving their uniqueness and showing that conditions (2.2) are necessary and sufficient for the isovariational forms to coincide with their adjoint. The novelty of conditions (2.2) is illustrated by the following

Corollary 1.A

Systems of ordinary isodifferential equations which are variationally iso-self-adjoint in isospace are generally variational non-self-adjoint when projected in ordinary spaces.

In fact, conditions (2.2) imply no restrictions on the isotopic terms $\hat{T}_k^{\;i}$ in isospace while the same terms are restricted by the ordinary conditions of self-adjointness in conventional spaces.

Theorem 2

The isotopic Newton equations (1.3.5) *are variationally iso-self-adjoint.*

Proof. The verification of the first set of conditions (2.2a) reads

$$\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^j} - \frac{\hat{\partial}\hat{F}_j}{\hat{\partial}\hat{a}^i} = \hat{T}_j^{\;m}\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^m} - \hat{T}_i^{\;m}\frac{\hat{\partial}\hat{F}_j}{\hat{\partial}\hat{a}^m} =$$

$$= \hat{T}_j^{\;m}\hat{T}_i^{\;m} - \hat{T}_i^{\;m}\hat{T}_j^{\;m} \equiv 0, \tag{2.3}$$

and the same identities hold for all remaining conditions. Q.E.D.

It is an instructive exercise for the interested reader to work out the isotopies of the remaining theorems for second-order ordinary differential equations (see [14], Sections 2.2 and 2.3).

We now introduce the conditions of variational iso-self-adjointness for N-dimensional systems (4.1) in an equivalent 2N-dimensional first-order form. Let $T^*\hat{E}(\hat{x},\hat{\delta},\hat{R})$ be the isocotangent space (seethe next paper for a geometric treatment) which in this section can be characterized via the independent space $\hat{E}(\hat{p},\hat{\delta},\hat{R})$ with new, independent, covariant coordinates $\hat{p}_k$. Let the total representation space be $\hat{T}(t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$ with local chart $\hat{b} = (b^\mu) = (\hat{x}^k, \hat{p}_k)$, $\mu = 1, 2, ..., 2N$, $k = 1, 2, ..., N$. Assign sufficiently smooth and invertible prescriptions for the characterization of the new variables $\hat{p}_k$

$$\hat{p}_k = \hat{g}_k(\hat{t}, \hat{x}, \hat{v}), \tag{2.4}$$

with unique system of implicit functions $v^k = f^k(\hat{t}, \hat{x}, \hat{p})$

(see [14], Sect. 2.4, for the conventional case). By using the latter implicit functions, system (2.1) can be written in the equivalent 2N-dimensional form

$$\hat{F}_\mu(\hat{t}, \hat{b}, \hat{c}) = \hat{C}_{\mu\nu}(\hat{t}, \hat{b})\, \hat{c}^\nu + \hat{D}_\mu(\hat{t}, \hat{b}) = 0,$$

$$\hat{c}^\nu = \hat{\partial}\, \hat{b}^\nu / \hat{\partial}\hat{t}. \qquad (2.5)$$

Theorem 3

A necessary and sufficient condition for system (2.5) *which is at least twice isodifferentiable and regular* (det. $(\hat{C}_{\mu\nu})(\hat{D}) \neq 0$) *in a* (6N+1)*-dimensional region* $\hat{D}$ *of points* $(\hat{t}, \hat{b}, \hat{c}, \hat{\partial}\hat{c}/\hat{\partial}\hat{t})$ *to be iso-self-adjoint in* $\hat{R}$ *is that all the following conditions*

$$\hat{C}_{\mu\nu} + \hat{C}_{\nu\mu} = 0, \qquad (2.6.a)$$

$$\frac{\partial \hat{C}_{\mu\nu}}{\partial \hat{b}^\rho} + \frac{\partial \hat{C}_{\nu\rho}}{\partial \hat{b}^\mu} + \frac{\partial \hat{C}_{\rho\mu}}{\partial \hat{b}^\nu} = 0, \qquad (2.6b)$$

$$\frac{\partial \hat{D}_\mu}{\partial \hat{b}^\nu} + \frac{\partial \hat{D}_\nu}{\partial \hat{b}^\mu} = \frac{\partial \hat{C}_{\mu\nu}}{\partial \hat{t}}, \qquad (2.6c)$$

are identically satisfying in $\hat{D}$.

The proof is also a simple isotopy of the proof of Theorem 2.7.2, p. 87, ref. [14]. Also, conditions (2.6) are uniquely derivable from conditions (2.2) when systems (2.1) are assumed to be 2N-dimensional and of first-order. The following property is self-evident,

COROLLARY 3.A: *When systems* (2.5) *assume the isocanonical form*

$$\hat{F}_\mu(\hat{t}, \hat{b}, \hat{c}) = \omega_{\mu\nu}\, \hat{c}^\nu - \hat{\Xi}_\mu(\hat{t}, \hat{b}) = 0, \qquad (2.7)$$

where $\omega_{\mu\nu}$ *is the conventional canonical symplectic tensor* (1.5) *the conditions of variational iso-self-adjointness* (2.6) *reduce to*

$$\frac{\partial \hat{\Xi}_\mu}{\partial \hat{b}^\nu} - \frac{\partial \hat{\Xi}_\nu}{\partial \hat{b}^\mu} = 0. \qquad (2.8)$$

Note that a conventional canonical system which is self-adjoint is also iso-self-adjoint, and this illustrates the reason why the potential representation of a selfadjoint forces persists at the isotopic level. Additional properties of variational iso-self-adjointness will be identified later on.

Let us recall the following meanings of the conditions of variational self-adjointness for 2N-dimensional systems of ordinary first-order differential equations (2.5) [14,15]:

1) *Analytic meaning*. The conditions imply the direct derivability (i.e., derivability without change of local variables or integrating factors) of the equations from a first-order variational principle

$$\delta A = \delta \int_{t_1}^{t_2} dt\, [\, R_\mu(t, b)\, db^\mu - H(t, b)\,] = 0, \qquad (2.9a)$$

$$C_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu, \quad D_\mu = \partial_\mu H - \partial_t R,$$

$$\partial_\mu = \partial / \partial b^\mu, \ \partial_t = \partial / \partial t; \qquad (2.9b)$$

2) *Geometric meaning*. The two form

$$C = C_{\mu\nu} db^\mu \wedge db^\nu, \qquad (2.10)$$

characterized by the covariant tensor $C_{\mu\nu}(b)$ is symplectic; and

3) *Algebraic meaning*. The brackets among two smooth functions A(b) and B(b) on the cotangent bundle

$$[A, B] = (\partial_\mu A)\, C^{\mu\nu}(b)\, (\partial_\nu B), \quad C^{\mu\nu} = [\,(C_{\alpha\beta})^{-1}\,]^{\mu\nu}, \qquad (2.11)$$

are Lie.

In the next sections we show that the above properties persist when formulated under isotopies in isospaces.

3. Isotopies of Lagrangian mechanics.

We now show the derivability of the isotopic Newton equations from a first-order iso-variational principle and then study the isotopies of Lagrange's [8] and Hamilton's [6] mechanics.

Proposition 1

All Newtonian action functionals of second or *higher order in Euclidean space* $E(t)\times E(x,\delta,R)\times E(v,\delta,R)$ *whose integrand is sufficiently smooth and regular in a region* D *of their variables can always be identically rewritten as first-order action isofunctionals in isospace* $\hat{E}(t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta}\hat{R})$ *which are bilinear in the velocities,*

$$\hat{A} = \int_{t_1}^{t_2} dt\, \mathcal{L}(t, x, v, a, ...) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \hat{L}(\hat{t}, \hat{x}, \hat{v}), \quad (3.1a)$$

$$\hat{L} = \tfrac{1}{2}\, \hat{m}\, \hat{v}^i \hat{\delta}_{ij} \hat{v}^j - \hat{U}^i(\hat{t}, \hat{x})\, \hat{\delta}_{ij}\, v^j - U_o(\hat{t}, \hat{x}) =$$

$$\tfrac{1}{2}\, \hat{m}\, \hat{v}_k \hat{v}^k - \hat{U}_k(\hat{t}, \hat{x})\, v^k - \hat{U}_o(\hat{t}, \hat{x}), \quad (3.1b)$$

In fact, identities (3.1a) are overdetermined because, for each given $\mathcal{L}$, there exist infinitely many choices of $\hat{m}$, $\hat{T}^o{}_o$, $\hat{T}_i^j$, $\hat{U}_k$ and $\hat{U}_o$. We shall assume that integral terms are admitted in the integrand provided that they are all embedded in the isometric.

The *isovariational calculus* is a simple extension of the isodifferential calculus. In fact, we can write the following isovariation along an admissible isodifferentiable path $\hat{P}$

$$\hat{\delta}\, \hat{A}(\hat{P}) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \{ \hat{\delta}\hat{x}^k \frac{\hat{\partial}}{\hat{\partial}\hat{x}^k} + \hat{\delta}\hat{v}^k \frac{\hat{\partial}}{\hat{\partial}\hat{v}^k} \}\, \hat{L}(\hat{P}) =$$

$$\hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{v}^k} - \frac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{x}^k} \} (\hat{P})\, \hat{\delta}\hat{x}^k, \quad (3.2)$$

where we have used isointegration by parts. The isotopy of the celebrated Euler [5] necessary condition can be formulated as follows.

Theorem 4 (Isoeuler necessary condition)

A necessary condition for an isodifferentiable path $\hat{P}_o$ *in isospace* $\hat{E}(t),\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta},\hat{R})$ *to be an extremal of action isofunctional* $\hat{A}$ *is that all the following isotopic equations*

$$\hat{L}_k(\hat{P}_o) = \{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\, \hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\, \hat{v}^k} - \frac{\hat{\partial}\, \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\hat{\partial}\, \hat{x}^k} \} (\hat{P}_o) = 0, \quad (3.3)$$

are identically verified along $\hat{P}_o$.

It is an instructive exercise for the interested reader to prove the following:

Corollary 4.A

Isotopic equations (5.3) are variationally iso-self-adjoint.

The isotopies of the remaining aspects of the calculus of variations (see, e.g., Bliss [4]) with consequential isotopies of the optimal control theory are intriguing and significant, but they cannot be studied here for brevity. When dealing with the calculus of isovariations, Eq.s (3.3) will be referred to as *isoeuler equations*, and when dealing with analytic mechanics they will be referred to as *isolagrange equations*.

We shall say that the isotopic Newton equations (1.3.5) admit a *direct isoanalytic representation* when there exists one isolagrangian $\hat{L}(\hat{t}, \hat{x}, \hat{v})$ and related isounit $\hat{1}$ under which all the following identities occur

$$\{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\, \hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\, \hat{v}^k} - \frac{\hat{\partial}\, \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\hat{\partial}\, \hat{x}^k} \}^{ISOSA} =$$

$$=\{ \hat{m} \frac{\hat{d}v_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\, \hat{U}_k(\hat{t}, \hat{x})}{\hat{\partial}\, \hat{x}^i} \frac{\hat{d}\, \hat{x}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_o(\hat{t}, \hat{x})}{\hat{\partial}\, \hat{x}^k} \}^{ISOSA} =$$

$$= \hat{T}_k{}^i \{ m \frac{dv_i}{dt} - \frac{\partial\, U_i(t, x)}{\partial x^s} \frac{dx^s}{dt} + \frac{\partial\, U_o(t, x)}{\partial x^i} -$$

$$- F^{NSA}{}_i(t, x, v) \}^{NSA} = 0. \quad (3.4a)$$

$$\hat{L}(\hat{t}, \hat{x}, \hat{v}) = \tfrac{1}{2}\, \hat{m}\, \hat{v}^k \hat{v}_k - \hat{U}, \quad \hat{U}(\hat{t}, \hat{x}, \hat{v}) =$$

$$= \hat{U}_k(\hat{t}, \hat{x})\, \hat{v}^k + \hat{U}_o(\hat{t}, \hat{x}), \quad (3.4b)$$

Theorem 5 (Universality of the isolagrangian mechanics)

All possible sufficiently smooth and regular dynamical systems in a star-shaped neighborhood of a point of their variables always admit a direct isorepresentation via the isolagrange equations in isospace.

Proof. The universality of the isorepresentation follows from the fact that conditions (I.3.8) always admit solution (I.3.10) in the unknown functions. **q.e.d.**

Remark. Newtonian systems are usually referred to systems with local-differential forces depending at most on velocities. Theorem 5 includes also non-Newtonian forces, e.g., forces of integral type or acceleration-dependent. Discontinuous Newtonian forces, such as those of impulsive type, have been removed from the theorem because of lack of current knowledge on the topology of isospaces with discontinuous isounits (isospaces of Kadeisvili's Class V, see Paper I), although such an extension is expected to exist, and its study is left to interested readers.

Note the simplicity of the construction of an isolagrangian representation as compared to the complexity of the construction of a conventional Lagrangian representation [14,15], when it exists.

4. Isotopies of Hamiltonian mechanics.

We now introduce, apparently for the first time, the isotopies of the Legendre transform based on the isodifferential calculus. For this purpose, we introduce the following isodifferentials in isospace $\hat{S}(t, \hat{x}, \hat{p}) = \hat{E}(t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$

$$\hat{\partial}\hat{t} = \hat{1}^o{}_o\, dt, \quad \hat{\partial}\hat{x}^k = \hat{1}^k{}_i\, dx^i, \quad \hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}^j = \delta^i{}_j, \text{ etc.}, \tag{4.1a}$$

$$\hat{\partial}\hat{p}_k = \hat{T}_k{}^i\, d\hat{p}_i, \quad \hat{\partial}\hat{p}^k = \hat{1}^k{}_i\, d\hat{p}^i, \tag{4.1b}$$

$$\hat{\partial}\hat{p}_i / \hat{\partial}\hat{p}_j = \delta_i{}^j, \text{ etc.} \tag{4.1c}$$

The total isounits and isotopic elements of the isospace $\hat{S}(t,\hat{x},\hat{p}) = \hat{E}(t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$ are therefore given by

$$\hat{1}_2 = \text{diag.}(\hat{1}^o{}_o, \hat{1}, \hat{T}), \quad \hat{T}_2 = \text{diag.}(\hat{T}_o{}^o, \hat{T}, \hat{1}). \tag{4.2}$$

It should be indicated that, in view of the independence of the variables $\hat{p}_k$ from $\hat{x}^k$, we can introduce a new isounit $\hat{W} = \hat{Z}^{-1}$ for the isospace $\hat{E}(\hat{p},\hat{\delta},\hat{R})$ which is different than the unit $\hat{1} = \hat{T}^{-1}$ of isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R})$, in which case the total unit is $\hat{1}_2 = \text{diag.}(\hat{1}^o{}_o, \hat{1}, \hat{W})$. Selection (4.1b) is the simplest possible case with $\hat{W} = \hat{1}$ which is recommendable for the geometric isotopies studied in the next paper. Other alternatives belong the the problem of the degrees of freedom of the isotopic theories which is not studied at this time for brevity.

We now introduce the *isocanonical momentum* via the following realization of prescriptions (2.4)

$$\hat{p}_k = \frac{\hat{\partial}\hat{L}(t, \hat{x}, \hat{v})}{\hat{\partial}\hat{v}^k} = \hat{m}\,\hat{v}_k - \hat{U}_k(t, \hat{x}), \tag{4.3}$$

under the condition of being regular in a (2N+1)-dimensional region $\hat{D}$ of points $(t, \hat{x}, \hat{p})$

$$\text{Det.}\left(\frac{\hat{\partial}^2\hat{L}(t, \hat{x}, \hat{v})}{\hat{\partial}\hat{v}^i\hat{\partial}\hat{v}^j}\right)(\hat{\mathcal{R}}) \neq 0. \tag{4.4}$$

thus admitting a unique set of implicit functions $\hat{v}^k = f^k(t, \hat{x}, \hat{p})$. The *isolegendre transform* can then be defined by

$$\hat{L}(t, \hat{x}, \hat{v}(t, \hat{x}, \hat{p}) = \hat{p}_k\,\hat{v}^k(t, \hat{x}, \hat{p}) - \tfrac{1}{2}\hat{m}\,\hat{v}_i(t, \hat{x}, \hat{p})\,\hat{v}^i(t, \hat{x}, \hat{p}) +$$

$$+ \hat{U}_k(t, \hat{x})\,\hat{v}^k(t, \hat{x}, \hat{p}) + \hat{U}_o(t, \hat{x}) = \hat{p}_k\,\hat{p}^k / 2\hat{m} +$$

$$+ \hat{V}^k(t, \hat{x})\,\hat{p}_k + \hat{V}^o(t, \hat{x}) = \hat{H}(t, \hat{x}, \hat{p}). \tag{4.5}$$

We are now equipped to study the isotopies of Hamilton's principle [6]. By using the unified variables $\hat{b} = \{\hat{b}^\mu\} = \{\hat{x}^k, \hat{p}_k\}$, $\hat{c}^\mu = \hat{\partial}\hat{b}^\mu / \hat{\partial}\hat{t}$, and by introducing the notation

$$\hat{R}^o = \{\hat{R}^o{}_\mu\} = \{\hat{p}_k, \hat{0}\}, \quad \mu = 1, 2, ..., 2N,$$

$$k = 1, 2, ..., N, \tag{4.6}$$

the *isocanonical principle* assumes the form along an actual path $\hat{P}_o$

$$\hat{\delta}\hat{A}^o = \hat{\delta}\hat{\int}_{t_1}^{t_2} \hat{\partial}\hat{t}\,(\hat{p}_k\,\hat{\partial}\hat{x}^k/\hat{\partial}\hat{t} - \hat{H})\,(\hat{P}_o) =$$

$$= \hat{\delta}\hat{\int}_{t_1}^{t_2} \hat{\partial}\hat{t}\,(\hat{R}^o{}_\mu\,\hat{c}^\mu - \hat{H})\,(\hat{P}_o) =$$

$$= \hat{\int}_{t_1}^{t_2} \hat{\partial}\hat{t}\,\{\,\hat{\delta}\hat{p}_i\frac{\hat{\partial}}{\hat{\partial}\hat{p}_i} + \hat{\delta}\hat{v}^i\frac{\hat{\partial}}{\hat{\partial}\hat{v}^i} + \hat{\delta}\hat{x}^i\frac{\hat{\partial}}{\hat{\partial}\hat{x}^i}\,\}\,(p_k\,v^k - H)\,(\hat{P}_o) =$$

$$\int_{\hat t_1}^{\hat t_2} \hat{d}t\,[\,(\frac{\hat d\hat x^k}{\hat d t}\frac{\partial \hat p_k}{\partial \hat p_i} - \frac{\partial \hat H}{\partial \hat p_i})\,\delta\hat p_i -$$

$$-\,[\frac{\hat d}{\hat d t}(\hat p_k \frac{\partial \hat v^k}{\partial \hat v^i}) + \frac{\partial \hat H}{\partial \hat x^i})\,\delta \hat x^i\,]\,(\hat P_o) =$$

$$= \int_{\hat t_1}^{\hat t_2} \hat d t\,\{\,\delta \hat b^\mu \frac{\partial}{\partial \hat b^\mu} + \delta\,\hat c^\mu \frac{\partial}{\partial \hat c^\mu}\}\,(\hat R^\circ_\mu\,\hat d \hat b^\mu - \hat H\,\hat d t\,)\,(\hat P_o) =$$

$$= \int_{\hat t_1}^{\hat t_2} \{(\frac{\partial \hat R^\circ_\nu}{\partial \hat b^\mu} - \frac{\partial \hat R^\circ_\mu}{\partial \hat b^\nu})\,\frac{\hat d \hat b^\nu}{\hat d t} - \frac{\partial \hat H}{\partial \hat b^\mu}\}\,(\hat P_o) * \delta \hat b^\mu = 0\,, \tag{4.7}$$

Theorem 6 (Isohamilton Necessary condition)

A necessary condition for an isofunctional in isocanonical form whose integrand is sufficiently smooth and regular in a region $\hat D$ *of points* $(\hat t, \hat b, \hat c)$ *to have an extremum along a path* $\hat P_o$ *is that all the following isoequations in disjoint notation*

$$\frac{\hat d\hat x^k}{\hat d t} = \frac{\partial \hat H(\hat t, \hat x, \hat p)}{\partial \hat p_k}\,, \qquad \frac{\hat d \hat p_k}{\hat d t} = - \frac{\partial \hat H(\hat t, \hat x, \hat p)}{\partial \hat x^k}\,, \tag{4.8}$$

or in unified notation

$$(\frac{\partial \hat R^\circ_\nu}{\partial \hat b^\mu} - \frac{\partial \hat R^\circ_\mu}{\partial \hat b^\nu})\,\frac{\hat d \hat b^\nu}{\hat d t} - \frac{\partial \hat H(\hat t, \hat b)_A}{\partial \hat b^\mu} = 0\,, \tag{4.9}$$

hold along an actual path $\hat P_o$.

It is also instructive for the interested reader to prove the following:

Corollary 6.A

Isotopic equations (4.9) *are variationally iso-self-adjoint.*

Eq.s (4.8) or (4.9) are called *isohamilton equations* and can be more simply written in the following covariant and contravariant forms, respectively,

$$\omega_{\mu\nu}\,\frac{\hat d \hat b^\nu}{\hat d t} = \frac{\partial \hat H(\hat t, \hat b)}{\partial \hat b^\mu}\,, \tag{4.10a}$$

$$\frac{\hat d \hat b^\mu}{\hat d t} = \omega^{\mu\nu}\,\frac{\partial \hat H(\hat t, \hat b)}{\partial \hat b^\nu}\,, \tag{4.10b}$$

where the quantities

$$(\,\omega_{\mu\nu}) = (\frac{\partial \hat R^\circ_\nu}{\partial \hat b^\mu} - \frac{\partial \hat R^\circ_\mu}{\partial \hat b^\nu}) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N}\end{pmatrix}, \tag{4.11a}$$

$$(\,\omega^{\alpha\beta}) = (\frac{\partial \hat R^\circ_\nu}{\partial \hat b^\mu} - \frac{\partial \hat R^\circ_\mu}{\partial \hat b^\nu})^{-1} = \begin{pmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N}\end{pmatrix}, \tag{4.11b}$$

are the *conventional* covariant and contravariant canonical tensors, respectively, which hold in view of the properties originating from Eq.s (4.1),

$$\partial \hat R^\circ_\nu / \partial \hat b^\mu \equiv \partial R^\circ_\nu / \partial b^\mu. \tag{4.12}$$

The equivalence of the isolagrangian and isohamiltonian equations under the assumed regularity and invertibility of the isolegendre transform can be proved as in the conventional case (see, e.g., [14], Sect. 3.8).

We now study the following additional property of isohamiltonian mechanics which is important for operator maps. The *isotopic Hamilton-Jacobi problem* (see, e.g., [15], p. 201 and ff. for the conventional case) is the identification of an isocanonical transform under which the Hamiltonian becomes null. The generating function of such a transform is the isocanonical action itself, resulting in the end-point contributions

$$\hat d\hat A = \hat d \int_{t_0}^{t} (\hat p_k\,\hat d\hat x^k - \hat H\,\hat d t\,) = \left|\,\hat p_k\,d\hat x^k - \hat H\,\hat d t\,\right|_{t_0}^{t}, \tag{4.13}$$

with *isotopic Hamilton-Jacobi equations*

$$\frac{\partial \hat A}{\partial \hat t} + \hat H(\hat t, \hat x, \hat p) = 0\,, \qquad \frac{\partial \hat A}{\partial \hat x^k} - \hat p_k = 0\,. \tag{4.14}$$

plus initial conditions $\partial\hat A/\partial\hat x^{\circ k} = \hat p_k^\circ$, where $\hat x^\circ$ and $\hat p^\circ$ are constants. The reader can easily work out the remaining properties of the isohamiltonian mechanics.

Remark 1. Note the *abstract identity between the conventional and isotopic mechanics.* Since the isounits are positive-definite, at the abstract level there is no distinction between dt and $\hat{d}t$, dx and $\hat{d}\hat{x}$, etc. The isolagrange and isohamilton equations therefore coincide at the abstract level with the conventional equations. This illustrates the axiom-preserving character of the isotopies, this time, at the analytic level.

Remark 2. The *direct universality* of the isohamiltonian mechanics for nonhamiltonian as well as nonlocal-integral systems in the fixed inertial frame of the observer should be compared with the corresponding *lack* of universality of the conventional Hamiltonian mechanics, as well as with the lack of applicability of Birkhoffian mechanics for nonlocal-integral systems, as discussed in Sect. 1.

Remark 3. The connection between the Birkhoffian and the isohamiltonian mechanics is intriguing. In fact, the Pfaffian action can always be identically rewritten as the isotopic action

$$\int_{t_1}^{t_2} [R_\mu(b)\, db^\mu - H(t, b)\, dt\,] \equiv$$
$$\equiv \int_{\hat{t}_1}^{\hat{t}_2} [R^\circ_\mu(b)\, \hat{d}\hat{b}^\mu - \hat{H}(\hat{t}, \hat{b})\, dt\,], \quad \hat{b}^\mu \equiv b^\mu \;\; \hat{H} \equiv H,$$
$$\hat{d}t = dt, \tag{4.15}$$

and the general, totally antisymmetric Lie tensor $\Omega^{\mu\nu}$ always admits the factorization into the canonical Lie tensor $\omega^{\mu\nu}$ and a regular symmetric matrix $\hat{T}_\mu{}^\nu$

$$\Omega^{\mu\nu} \;\; \equiv \;\; \omega^{\alpha\beta}\, \hat{T}_\beta{}^\nu, \tag{4.16}$$

under which Birkhoff's equations (1.10) coincides with the isohamilton's equations (4.10b) for $\hat{1}_o{}^\circ = 1$. As a result, *Birkhoffian mechanics is a particular case of the isohamiltonian mechanics.*

Despite these similarities, it should be indicated that the isohamiltonian mechanics is considerably broader than the Birkhoffian mechanics. In fact, the former is based on an action of arbitrary order, while the latter necessarily requires a first-order action. Also, the former can represent integral forces, while the latter cannot (because the underlying geometry, the symplectic geometry in its most general possible exact realization) only admits local-differential systems. Finally, the former is based on a broader mathematics, the isodifferential calculus, while the latter is based on conventional mathematics.

Remark 4. Note that the isotopic Hamilton-Jacobi equations (4.14) imply the properties

$$\hat{\partial}\hat{A}^\circ / \hat{\partial}\hat{p}_k \equiv 0, \quad k = 1, 2, ..., N, \tag{4.17}$$

which are *necessary* for a correct isotopy of quantization studied in the next section and in the next paper (otherwise, the "wavefunctions" would depend also on the momenta, $\psi = \psi(\hat{t}, \hat{x}, \hat{p}$, thus being topologically inequivalent to the quantum mechanical wavefunctions $\psi(t, x)$).

By comparison, Pfaffian principle (1.9) implies the following *Birkhoffian Hamilton-Jacobi equations* (studied in detail in [15])

$$\frac{\partial A}{\partial t} + H(t, x, p) = 0, \tag{4.18a}$$

$$\frac{\partial A}{\partial x^k} - P_k(x, p) = 0, \tag{4.18b}$$

$$\frac{\partial A}{\partial p_k} - Q^k(x, p) = 0, \tag{4.18c}$$

for which $\partial A/\partial p_k \neq 0$. As a result, Birkhoffian mechanics is not a suitable classical foundation for the isotopies of quantum mechanics. This illustrates an additional reason why, after constructing the Birkhoffian generalization of Hamiltonian mechanics [15], this author had to search for an additional, more suitable generalization.

Remark 5. An important application of the isohamiltonian mechanics is to provide a novel classical realization of the Lie-Santilli isotheory (Paper I). Recall that the conventional classical realization of the Lie product is given by the familiar Poisson brackets among two functions A(b) and B(b) in the cotangent bundle,

$$[A, B]_{\text{Poisson}} = \frac{\partial A}{\partial x^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial x^k}\frac{\partial A}{\partial p_k} =$$

$$= \frac{\partial A}{\partial b^\mu}\, \omega^{\mu\nu}\, \frac{\partial B}{\partial b^\nu}. \tag{4.19}$$

From the self-adjointness of Birkhoff's equations [15] and the algebraic meaning of the conditions of self-adjointness recalled in Sect. 2, the most general possible (regular, unconstrained) brackets in the cotangent bundle verifying the Lie algebra axioms are given by the *Birkhoffian brackets* (also called generalized Poisson brackets) [15]

$$[A, B]_{Birkhoff} = \frac{\partial A}{\partial b^\mu}\, \Omega^{\mu\nu}(b)\, \frac{\partial B}{\partial b^\nu}, \quad \Omega^{\mu\nu} = [\,(\Omega_{\alpha\beta})^{-1}\,]^{\mu\nu} \tag{4.20}$$

The novel brackets introduced in this paper are given by the following brackets among isofunctions $\hat{A}(\hat{b})$, $\hat{B}(\hat{b})$

$$[A, B]_{Isotopic} = \frac{\partial A}{\partial \hat{x}^k}\frac{\partial B}{\partial \hat{p}_k} - \frac{\partial B}{\partial \hat{x}^k}\frac{\partial A}{\partial \hat{p}_k} =$$

$$= \frac{\partial A}{\partial \hat{x}^k}\frac{\partial B}{\partial \hat{p}_k} - \frac{\partial B}{\partial \hat{x}^k}\frac{\partial A}{\partial \hat{p}_k}, \tag{4.21}$$

and they *formally coincide* with the conventional brackets (4.19) when projected in the original space. However, one should remember that the underlying geometry is generalized. In fact, the isotopic brackets can be written

$$[A, B]_{Isotopic} = \frac{\partial A}{\partial \hat{x}_i}\hat{T}_i^{\,k}(t, r, p, ...)\,\delta_{kj}\,\frac{\partial B}{\partial \hat{p}_j}$$

$$- \frac{\partial B}{\partial \hat{x}_i}\,\hat{T}_i^{\,k}(t, r, p, ...)\,\delta_{kj}\,\frac{\partial A}{\partial \hat{p}_j}. \tag{4.22}$$

thus showing their differences with the conventional brackets. Moreover, one should keep in mind from the comments following Eq.s (4.1) that we have selected the simplest possible isotopies for which the isounits of the independent variables $\hat{p}_k$ and x^k are inverse of each other. The use of different isounits for $\hat{p}_k$ and x^k evidently implies further differences between the isotopic and conventional brackets.

Remark 6. Note that the Lie-isotopic character of brackets (4.19) is assured by the iso-self-adjointness of the isohamilton equations. Note also that brackets (4.20) *do not* verify the Lie algebra axioms in conventional spaces, evidently because the isotopic elements $\hat{T}_i^{\,j}$ are unrestricted. This illustrates that *the isotopic theory of this paper verifies the Lie axioms only in isospace but not when projected in conventional spaces.* This occurrence should be compared to other realizations studied in ref.s [15,19] in which the Lie axioms are verified in isospace as well as in their projection in conventional spaces.

Remark 7. It is also easy to see that the isohamiltonian mechanics provides a classical realization of the Lie-Santilli isogroups (see Paper I). In fact, the integrated form of Eq. (4.10b) yields the time evolution of a quantity $\hat{A}(t)$ in isospace

$$\hat{A}(t) = \exp\{\,t\,[\frac{\partial \hat{H}}{\partial \hat{x}}\hat{T}\frac{\partial}{\partial \hat{p}} - \frac{\partial \hat{H}}{\partial \hat{x}}\hat{T}\frac{\partial}{\partial \hat{p}}]\}\;\hat{A}(0), \tag{4.23}$$

which, for the case of a diagonal isotopic element, can be expressed in term of the isoexponentiation (see Paper I)

$$\hat{A}(t) = \{\,\hat{e}^{\,t\,\omega^{\mu\nu}(\partial_\mu H)\,\partial_\nu}\}\,\hat{T}\,\hat{A}(0) = \{\,\hat{e}^{\,t\,\omega^{\mu\nu}(\partial_\mu H)\,\hat{T}\,\partial_\nu}\}\,\hat{1}\,\hat{T}\,\hat{A}(0), \tag{4.24}$$

and it is a one-dimensional isogroup owing to the appearance of the isotopic matrix $\hat{T}$ in the exponent.

5. Isotopies of quantum mechanics.

The significance of isohamiltonian mechanics is also illustrated by the fact that its map under the conventional (or symplectic) quantization *is not* quantum mechanics, but instead a broader isotopic theory submitted by Santilli [13] under the name of *hadronic mechanics* and then studied by various authors (see the comprehensive presentations [19,20]). Without entering into details, it is important for this paper to see that the isotopic operator theory preserves all the main features of the isotopic Newton equations, such as the representation of nonspherical-deformable shapes, nonselfadjoint forces and nonlocal-integral interactions.

In this section we study the isotopies of the simplest possible quantization, called *naive quantization*, while those of the symplectic quantization (initiated by Lin [9] will be studied in the next section. The naive quantization is the map of the canonical action functional in Eq. (1.1) $A° \rightarrow -i\hbar \, Ln\psi(t, x)$, where $\hbar = 1$ is the unit of quantum mechanics, which maps the conventional Hamilton-Jacobi equations into Schrödinger's equations for the energy and momentum. Since the action $\hat{A}°$ is an isotopy of $A°$, the preceding map must also subjected to an isotopy. Animalu and Santilli [2] therefore introduced the following *naive isoquantization*

$$\hat{A}°(t, \hat{x}) \quad \rightarrow \quad -i\hat{1}(t, \hat{p})\, Ln\, \hat{\psi}(t, \hat{x}), \quad \hbar = 1, \qquad (5.1)$$

where the coordinates are in isospace, $\hat{1}$ is the isounit of the isotopic Newton equations which is here assumed to be independence from $\hat{x}$ for simplicity (see ref. [20] for the general case). The application of map (5.1) to Eq.s (4.14) yields: the *isoschrödinger equation in the energy*

$$i\, \partial \hat{\psi} / \partial t = \hat{H} \hat{T} \hat{\psi} = \hat{H} * \hat{\psi}, \qquad (5.2)$$

first introduced by Myung and Santilli [10] in terms of conventional differential calculus, and formulated for the first time here for the isodifferential calculus; the *isoschrödinger equation in the momentum*

$$\hat{p}_k \hat{T} \hat{\psi} = \hat{p}_k * \hat{\psi} = -i\, \partial \hat{\psi} / \partial \hat{x}^k, \qquad (5.3)$$

first introduced by Santilli [16]; and the related *fundamental isocommutation rules*

$$[\hat{b}_\mu \hat{,} \hat{b}_\nu] = \hat{b}_\mu \hat{T} \hat{b}_\nu - \hat{b}_\nu \hat{T} \hat{b}_\mu = \omega_{\mu\nu} \hat{1},$$

$$\hat{b} = \{\hat{x}_k, \hat{p}_k\} \qquad (5.4)$$

also originally due to Santilli [16]. Note the preservation of the *conventional* symplectic structure $\omega_{\mu\nu}$.

The emerging operator structure is characterized by:

1) Enveloping operator algebra $\hat{\xi}$ with generic elements $\hat{A}$, $\hat{B}$, ... (which are polynomials in $\hat{x}$ and $\hat{p}$) called *isoassociative envelope* because characterized by the isoassociative product $\hat{A} * \hat{B} = \hat{A}\hat{T}\hat{B}$ with isounit $\hat{1} = \hat{T}^{-1}$ originally due to Santilli [13];

2) The isofields $\hat{C}(\hat{c},+,\hat{x})$ of isocomplex numbers $\hat{c}$, or its isoreal particularization $\hat{R}(\hat{n},+,\hat{x})$ (see paper I); and

3) The *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $\hat{\psi}$, $\hat{\phi}$, ..., and *isoinner product* over $\hat{C}$

$$<\hat{\psi} \hat{\uparrow} \hat{\phi}> = \hat{1} \int d\hat{x}^3 \hat{\psi}^\dagger \hat{\phi} \in \hat{C}(\hat{c},+,\hat{x}), \qquad (5.5)$$

originally submitted by Myung and Santilli [10] (see [19] for recent studies).

The isotopies of the Heisenberg representation then yield the *isoheisenberg equation* for an observable $\hat{O}$

$$i\, \partial \hat{O} / \partial t = [\hat{O} \hat{,} \hat{H}] = \hat{O} * \hat{H} - \hat{H} * \hat{O} = \hat{O}\hat{T}\hat{H} - \hat{H}\hat{T}\hat{O} \qquad (5.6)$$

originally submitted by Santilli [12,13]. The operator image of the isobrackets (4.21) is therefore given by

$$[\hat{A} \hat{,} \hat{B}] = \hat{A}\hat{T}\hat{B} - \hat{B}\hat{T}\hat{A}, \qquad (5.7)$$

which constitute the operator realization of the Lie-Santilli isoalgebra (see again Paper I for references).

The exponentiated form of Eq.s (5.7) yields the time evolution of isostates in terms of isoexponentiations

$$\hat{\psi}' = \hat{U} * \hat{\psi} = \{\hat{e}^{i\hat{H}t}\} * \hat{\psi} = e^{iH\hat{T}t}\} \hat{\psi}, \qquad (5.8)$$

thus resulting to be an operator realization of the Lie-Santilli isogroups with laws

$$\hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{U}(\hat{0}) = \hat{1}. \qquad (5.9)$$

In particular, $\hat{H}$ is *isohermitean* and $\hat{U}$ is *isounitary* on $\hat{\mathcal{H}}$, i.e., it verifies the laws,

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{1}. \qquad (5.10)$$

In this paper we have shown that the isotopies are nonlinear, nonlocal and noncanonical maps of a conventional linear, local and canonical theory. To illustrate the axiom-preserving character of the maps, it remains to show that linearity, locality and canonicity is regained in isospaces over isofields.

The regaining of linearity in isospace, called *isolinearity*, is readily established by the fact that the isotransformations $\hat{x}' = \hat{A} * \hat{x} = \hat{A}\hat{T}\hat{x}$, $\hat{A} \in \hat{\xi}$ do indeed verify the condition in isospace

$$\hat{A} * (\hat{n} * \hat{x} + \hat{m} * \hat{y}) =$$

$$\hat{n} * (\hat{A} * \hat{x}) + \hat{m} * (\hat{A} * \hat{y}), \quad \forall \hat{n}, \hat{m} \in \hat{R}, \ \hat{A} \in \hat{\xi}, \quad (5.11)$$

while their projection in the original space is nonlinear, e.g., $x' = \hat{A}T(x, \ldots)x$. As a result, the theory of isooperators on the isohilbert space $\hat{\mathcal{H}}$ over $\hat{C}$ is also isolinear. The regaining of locality in isospace, called *isolocality*, is established by the fact that the theory is everywhere local except at the unit. Finally, the regaining of canonicity in isospace, called *isocanonicity*, is established by the fact that, e.g., the isoaction $\hat{A}^\circ$ coincides with the canonical action A° at the abstract level.

In summary, the matrix $\hat{T}$ of the isotopic Newton equations of Paper I is preserved in its entirety at the operator level. This confirms the capability of the isotopies of quantum mechanics (hadronic mechanics [13]) of representing nonspherical-deformable shapes, nonselfadjoint forces and nonlocal-integral interactions (see [20] for comprehensive studies and applications to nuclear physics, particle physics, astrophysics, superconductivity and other fields).

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