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A generalization of the integral operators involving the Gauss' hypergeometric function

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Abstract

In this paper, integral operators $I_x^{\alpha,\beta,\eta,\mu} f$ and $J_x^{\alpha,\beta,\eta,\mu} f$, which involve the Gauss' hypergeometric function and generalizing Saigo's operators $I_x^{\alpha,\beta,\eta} f$ and $J_x^{\alpha,\beta,\eta} f$ respectively, are studied. Their properties and various composition rules are also presented. Some known results are obtained as particular cases.

Key words: Integral operators, Gauss' hipergeometric function.

Una generalización de los operadores integrales que involucran la función hipergeométrica de Gauss

Resumen

En el presente trabajo se estudian los operadores integrales $I_x^{\alpha,\beta,\eta,\mu} f$ y $J_x^{\alpha,\beta,\eta,\mu} f$ los cuales involucran la función hipergeométrica de Gauss, y generalizan los operadores de Saigo $I_x^{\alpha,\beta,\eta} f$ y $J_x^{\alpha,\beta,\eta} f$ respectivamente. Se presentan algunas propiedades y varias reglas de composición. Como casos particulares se obtienen algunos resultados conocidos.

Palabras claves: Operadores integrales, función hipergeométrica de Gauss.

Introducción

Los operadores de integración fraccional desempeñan un rol muy importante en el cálculo fraccional. Son de gran importancia por sus aplicaciones en la solución de ecuaciones diferenciales, integrales e integro-diferenciales. Los operadores más conocidos son los de Riemann-Liouville [1], Weyl [2], Erdélyi-Kober [3], Saigo [4,5], entre otros.

Además de los operadores de integración fraccional ya mencionados existen otros, los cuales han sido definidos por diversos autores [6], entre los más recientes podemos mencionar:

Los operadores de integración fraccional que involucran la función G de Meijer como núcleo. Estos operadores fueron definidos por V. Kiryakova [7-9]. Los operadores de integración fraccional que involucran la función H de Fox [10]. Kalla [11-12], introdujo dos operadores de integración fraccional que involucran la función H de Fox como núcleo.

Generalizaciones de operadores de integración fraccional han sido obtenidas por Erdélyi [13], Kober [14], Erdélyi-Kober [3] y Kalla [11,12, 15-17].

Los operadores de Saigo [4,5], están definidos por

$$I_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}) f(t)\, dt \tag{1}$$

y

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} .$$

$$F(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}) f(t) dt \tag{2}$$

$\alpha > 0$, β y η números reales

donde ${}_2F_1(\alpha,\beta;\gamma;z)$ denota la serie hipergeométrica de Gauss.

En este trabajo se generalizan estos operadores y se presentan algunas propiedades y varias reglas de composición. Como casos particulares se obtienen algunos resultados conocidos.

Generalización de los operadores de Saigo

Operador Integral $I_x^{\alpha,\beta,\eta,\mu} f$

Se define el operador generalizado de Saigo $I_x^{\alpha,\beta,\eta,\mu} f$, mediante la expresión:

$$I_x^{\alpha,\beta,\eta,\mu} f = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^x t^{\mu} (x-t)^{\alpha-1} .$$

$$F(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{x}) f(t)\, dt , \tag{3}$$

$\alpha > 0$, $\mu > -1$, β y η números reales.

Casos Particulares:

i) Cuando $\mu = 0$,

$$I_x^{\alpha,\beta,\eta,0} f = I_x^{\alpha,\beta,\eta} f \tag{4}$$

donde $I_x^{\alpha,\beta,\eta} f$ es el operador de Saigo definido en (1).

ii) Cuando $\mu = 0$ y $\alpha + \beta = 0$.

$$I_x^{\alpha,-\alpha,\eta,0} f = R_x^{\alpha} f \tag{5}$$

donde $R_x^{\alpha} f$ es el operador de Riemann-Liouville [4].

iii) Cuando $\mu = 0$ y $\beta = 0$,

$$I_x^{\alpha,0,\eta,0} f = E_x^{\alpha,\eta} f \tag{6}$$

siendo $E_x^{\alpha,\eta} f$ el operador de Erdélyi-Kober [4].

Algunas Propiedades del Operador $I_x^{\alpha,\beta,\eta,\mu} f$

Aplicando la relación

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z), \tag{7}$$

se demuestran las siguientes propiedades:

$$I_x^{\alpha,\beta,\eta,\mu} x^{\mu+\beta-\eta} f = I_x^{\alpha,\eta-\mu,\beta+\mu,\mu} f \tag{8}$$

$$I_x^{\alpha,\beta,\eta,\mu} f = x^{-\alpha-\beta-\eta-\mu} I_x^{\alpha,-\alpha-\eta-\mu,-\alpha-\beta-\mu,\mu} f \tag{9}$$

Reglas de Composición para el Operador $I_x^{\alpha,\beta,\eta,\mu} f$

Con las condiciones de la definición (3), se tienen las siguientes reglas de composición:

$$I_x^{\alpha,\beta,\eta,\mu} f = I_x^{\gamma,\delta,\eta,\mu} I_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu} f , \tag{10}$$

$\alpha > \gamma > 0$, $\mu > -1$.

Demostración

De (3),

$$I_x^{\gamma,\delta,\eta,\mu} I_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu} f = \frac{x^{-\gamma-\delta-2\mu}}{\Gamma(\gamma)\Gamma(\alpha-\gamma)} \int_0^x t^{-\alpha-\beta+\gamma+\delta} .$$

$$(x-t)^{\gamma-1} F(\gamma+\delta+\mu,-\eta;\gamma;1-\frac{t}{x}) \int_0^t s^{\mu}(t-s)^{\alpha-\gamma-1} .$$

$$F(\alpha+\beta-\gamma-\delta,-\gamma-\eta;\alpha-\gamma;1-\frac{s}{t}) f(s) ds dt$$

Intercambiando el orden de integración, en base a la convergencia absoluta, haciendo el cambio de variable $u = \frac{x-t}{x-s}$ y aplicando la fórmula [18]:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}(1-tz)^{-a'} .$$

$$F(a-a',b;\lambda;tz)\, F(a',b-\lambda;c-\lambda;\frac{z(1-t)}{1-tz})dt, \quad (11)$$

$Re(c) > \mathrm{Re}(\lambda) > 0, \quad |arg(1-z)| < \pi,$

se obtiene $I_x^{\alpha,\beta,\eta,\mu} f$.

De manera análoga, se demuestra que:

$$I_x^{\alpha,\beta,\eta,\mu} f = I_x^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta,\mu}\, I_x^{\gamma,\delta-\mu,\eta-\beta+\delta-\mu,\mu} f, \quad (12)$$

$\alpha > \gamma > 0,\ \mu > -1$.

Aplicando (10) y (12) se comprueba que:

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\alpha+\eta,\mu} f = I_x^{\alpha+\gamma,\beta+\delta+\mu,\eta,\mu} f, \quad (13)$$

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\eta-\beta-\gamma-\delta,\mu} f = I_x^{\alpha+\gamma,\beta+\delta+\mu,\eta-\gamma-\delta+\mu,\mu} f \quad (14)$$

Otras Fórmulas para $I_x^{\alpha,\beta,\eta,\mu} f$

Consideremos la transformada de Mellin:

$$M\{\varphi(x);z\} = \int_0^\infty x^{z-1}\varphi(x)\, dx,$$

donde z es una variable compleja.

Lema

Para $\mathrm{Re}(c) > 0$, $Re(f) > 0$, $\mathrm{Re}(z) > \max\{\mathrm{Re}(-p), \mathrm{Re}(a+b-c-p), \mathrm{Re}(-d+f-p+q), \mathrm{Re}(e-p+q)\}$ se cumple:

$$M\{W^{c+f-1} (1-W)^{d-f+p-q} \int_0^1 V^{c-1} (1-V)^{f-1}(1-VW)^{q-d} .$$
$$F(a,b;c;VW)F(d,e;f;\frac{W(1-V)}{1-VW})\, dV;z\} =$$
$$\Gamma(c)\Gamma(f)\, \frac{\Gamma(z+p)\Gamma(z-a-b+c+p)\,\Gamma(z+d-f+p-q)\Gamma(z-e+p-q)}{\Gamma(z-a+c+p)\Gamma(z-b+c+p)\,\Gamma(z+p-q)\Gamma(z+d-e+p-q)}, \quad (15)$$

Teorema

Si

$$I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;W) = \frac{W^{\alpha+\gamma-1}}{\Gamma(\alpha)\Gamma(\gamma)} .$$
$$\int_0^1 V^{\alpha-1}(1-V)^{\gamma-1}(1-WV)^{-\gamma-\delta-\mu} F(\alpha+\beta+\mu,-\eta;\alpha;WV).$$
$$F(\gamma+\delta+\mu,-\zeta;\gamma;\frac{W(1-V)}{1-WV})\, dV, \quad (16)$$

entonces

$$M\{I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;W);z\} = \frac{\Gamma(z)\Gamma(z-\beta+\eta-\delta-2\mu)}{\Gamma(z-\beta-\delta-2\mu)} .$$
$$.\ \frac{\Gamma(z-\delta+\zeta-\mu)}{\Gamma(z+\alpha+\eta-\delta-\mu)\Gamma(z+\gamma+\zeta)}, \quad (17)$$

$\alpha > 0,\ \gamma > 0, \mathrm{Re}(z) > máx\{0, \delta-\zeta+\mu, \delta+\mu, \beta-\eta+\delta+2\mu\}$.

Demostración

$$M\{I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;W);z\} = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} M\{W^{\alpha+\gamma-1} .$$
$$.\int_0^1 V^{\alpha-1}(1-V)^{\gamma-1}(1-WV)^{-\gamma-\delta-\mu} F(\alpha+\beta+\mu,-\eta;\alpha;WV) .$$
$$.\ F(\gamma+\delta+\mu,-\zeta;\gamma;\frac{W(1-V)}{1-WV})dV;z\} .$$

Aplicando (15) se obtiene lo requerido

$$M\{I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;W);z\} = \frac{\Gamma(z)\Gamma(z-\beta+\eta-\delta-2\mu)}{\Gamma(z-\beta-\delta-2\mu)\Gamma(z+\alpha+\eta-\delta-\mu)} .$$
$$.\ \frac{\Gamma(z-\delta+\zeta-\mu)}{\Gamma(z+\gamma+\zeta)}$$

De (17) y puesto que $M[f(x);s] = M[g(x);s]$ $\Longleftrightarrow f(x) = g(x)$, podemos establecer el siguiente lema.

Lema

Si $\alpha > 0,\ \gamma > 0$ y $0 < \lambda < \alpha + \gamma$, entonces se cumplen las identidades:

$$I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;W) =$$
$$= I(\lambda,-\lambda+\alpha+\beta,\eta,\mu;-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta,\mu;W) \quad (18)$$

$$= I(\lambda,-\lambda+\beta-\eta+\gamma+\delta+\zeta+\mu,\eta,\mu;-\lambda+\alpha+\gamma,\lambda+\eta-\gamma-\zeta-\mu,$$
$$\lambda+\eta-\gamma-\delta-\mu,\mu;W) \quad (19)$$

$$= I(\lambda,-\lambda+\alpha+\eta-\zeta-\mu,\beta+\zeta+\mu,\mu;-\lambda+\alpha+\gamma,\lambda-\alpha+\beta-\eta+\delta+\zeta+\mu,$$
$$\lambda-\alpha+\zeta,\mu;W) \quad (20)$$

$$= I(\lambda,-\lambda+\gamma+\delta,\beta+\zeta+\mu,\mu;-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,$$
$$\lambda+\eta-\gamma-\delta-\mu,\mu;W) \quad (21)$$

Utilizando el lema anterior, se verifican las siguientes igualdades:

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = I_x^{\lambda,-\lambda+\alpha+\beta,\eta,\mu} .$$
$$.\ I_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta,\mu} f \quad (22)$$

Demostración

Por definición

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)\Gamma(\gamma)}\,.$$

$$.\int_0^x t^{-\gamma-\delta-\mu}(x-t)^{\alpha-1}\, F(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{x})\,.$$

$$.\int_0^t s^{\mu}(t-s)^{\gamma-1}\, F(\gamma+\delta+\mu,-\zeta;\gamma,1-\frac{s}{t})\, f(s)\,ds\,dt\,.$$

Intercambiando el orden de integración, en base a la convergencia absoluta y haciendo el cambio de variable $u=\frac{x-t}{x-s}$, resulta:

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = \frac{x^{-\alpha-\beta-\gamma-\delta-3\mu}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^x s^{\mu}(x-s)^{\alpha+\gamma-1} f(s)\,.$$

$$.\int_0^1 u^{\alpha-1}(1-u)^{\gamma-1}[1-(1-\frac{s}{x})u]^{-\gamma-\delta-\mu} F[\alpha+\beta+\mu,-\eta;\alpha;(1-\frac{s}{x})u]$$

$$.\ F[\gamma+\delta+\mu,-\zeta;\gamma\,;\ \frac{(1-\frac{s}{x})(1-u)}{(1-(1-\frac{s}{x})u)}\ du\,ds\,.$$

Aplicando (16), se tiene:

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = x^{-\beta-\delta-3\mu-1}\,.$$

$$.\int_0^x s^{\mu} I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;1-\frac{s}{x})\, f(s)\,ds\,. \tag{23}$$

Por otro lado, aplicando (23) al lado derecho de (22) se tiene:

$$I_x^{\lambda,-\lambda+\alpha+\beta,\eta,\mu}\, I_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta,\mu} f = x^{-\beta-\delta-3\mu-1}\,.$$

$$.\int_0^x s^{\mu} I(\lambda,-\lambda+\alpha+\beta,\eta,\mu;-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta,\mu;1-\frac{s}{x}) f(s)\,ds,$$

de (18), resulta

$$I_x^{\lambda,-\lambda+\alpha+\beta,\eta,\mu}\, I_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta,\mu} f = x^{-\beta-\delta-3\mu-1}\,.$$

$$\int_0^x s^{\mu} I(\alpha,\beta,\eta,\mu;\gamma,\delta,\zeta,\mu;1-\frac{s}{x})\, f(s)\,ds = I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f\,.$$

Análogamente se demuestran las igualdades:

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = I_x^{\lambda,-\lambda+\beta-\eta+\gamma+\delta+\zeta+\mu,\eta,\mu}\,.$$

$$.\ I_x^{-\lambda+\alpha+\gamma,\lambda+\eta-\gamma-\zeta-\mu,\lambda+\eta-\gamma-\delta-\mu,\mu} f \tag{24}$$

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = I_x^{\lambda,-\lambda+\alpha+\eta-\zeta-\mu,\beta+\zeta+\mu,\mu}\,.$$

$$.\ I_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\beta-\eta+\delta+\zeta+\mu,\lambda-\alpha+\zeta,\mu} f \tag{25}$$

$$I_x^{\alpha,\beta,\eta,\mu}\, I_x^{\gamma,\delta,\zeta,\mu} f = I_x^{\lambda,-\lambda+\gamma+\delta,\beta+\zeta+\mu,\mu}\,.$$

$$.\ I_x^{-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\lambda+\eta-\gamma-\delta-\mu,\mu} f \tag{26}$$

Operador Integral $J_x^{\alpha,\beta,\eta,\mu} f$

Se define al Operador Generalizado de Saigo $J_x^{\alpha,\beta,\eta,\mu} f$, mediante la expresión

$$J_x^{\alpha,\beta,\eta,\mu} f = \frac{x^{-\mu}}{\Gamma(\alpha)} \int_x^{\infty} t^{-\alpha-\beta}(t-x)^{\alpha-1}\,.$$

$$F(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{x}{t}) f(t)\,dt, \tag{27}$$

$\alpha > 0$, $\beta > 0$, η y μ números reales.

Casos Particulares:

i) Cuando $\mu = 0$,

$$J_x^{\alpha,\beta,\eta,0} f = J_x^{\alpha,\beta,\eta} f, \tag{28}$$

donde $J_x^{\alpha,\beta,\eta} f$ es el operador de Saigo definido en (2).

ii) Cuando $\mu = 0$ y $\alpha + \beta = 0$,

$$J_x^{\alpha,-\alpha,\eta,0} f = W_x^{\alpha} f \tag{29}$$

donde $W_x^{\alpha} f$ es el operador de Weyl [4].

iii) Cuando $\mu = 0$ y $\beta = 0$,

$$J_x^{\alpha,0,\eta,0} f = K_x^{\alpha,\eta} f, \tag{30}$$

siendo $K_x^{\alpha,\eta} f$ el operador de Erdélyi-Kober [4].

Algunas Propiedades del Operador $J_x^{\alpha,\beta,\eta,\mu} f$

Aplicando (7) se demuestran las siguientes propiedades:

$$J_x^{\alpha,\beta,\eta,\mu}\, x^{\alpha+\beta+\eta+\mu} f = J_x^{\alpha,-\alpha-\eta-\mu,-\alpha-\beta-\mu,\mu} f \qquad (31)$$

$$J_x^{\alpha,\beta,\eta,\mu} f = x^{\eta-\beta-\mu}\, J_x^{\alpha,\eta-\mu,\beta+\mu,\mu} f, \qquad (32)$$

Usando (3), intercambiando el orden de integración en base a la convergencia absoluta y aplicando (27) se comprueba la propiedad

$$\int_0^\infty x^{2\mu} g(x)\, I_x^{\alpha,\beta,\eta,\mu} f dx = \int_0^\infty x^{2\mu} f(x)\, J_x^{\alpha,\beta,\eta,\mu} g dx, \qquad (33)$$

Reglas de Composición para el Operador $J_x^{\alpha,\beta,\eta,\mu} f$

Con las condiciones de la definición (27) se tienen las siguientes reglas de composición:

$$J_x^{\alpha,\beta,\eta,\mu} f = J_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu}\, J_x^{\gamma,\delta,\eta,\mu} f,$$
$$\alpha > \gamma > 0,\ \beta > \delta > 0\ . \qquad (34)$$

Demostración

De (27)

$$J_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu}\, J_x^{\gamma,\delta,\eta,\mu} f = \frac{x^{-\mu}}{\Gamma(\gamma)\Gamma(\alpha-\gamma)} \int_x^\infty t^{-\alpha+\gamma-\beta+\delta}\ .$$

$$.\ (t-x)^{\alpha-\gamma-1}\, F(\alpha-\gamma+\beta-\delta,-\gamma-\eta,\alpha-\gamma\,; 1-\frac{x}{t}) \int_t^\infty s^{-\gamma-\delta}(s-t)^{\gamma-1}\ .$$

$$F(\gamma+\delta+\mu,-\eta;\gamma\,; 1-\frac{t}{s})\, f(s) ds dt\ .$$

Intercambiando el orden de integración, en base a la convergencia absoluta y haciendo el cambio de variable $u = \frac{s-t}{s-x}$, se obtiene

$$J_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu}\, J_x^{\gamma,\delta,\eta,\mu} f = \frac{x^{-\mu}}{\Gamma(\gamma)\Gamma(\alpha-\gamma)} \int_x^\infty s^{-\alpha-\beta}(s-x)^{\alpha-1} .$$

$$.\ f(s) \int_0^1 u^{\gamma-1}(1-u)^{\alpha-\gamma-1}[1-(1-\frac{x}{s})\mu]^{-\alpha-\beta+\gamma+\delta}\ .$$

$$F[\alpha+\beta-\gamma-\delta,-\gamma-\eta;\alpha-\gamma\,; \frac{(1-\frac{x}{s})(1-u)}{(1-(1-\frac{x}{s})u)}]\ .$$

$$.\ F[\gamma+\delta+\mu,-\eta;\gamma\,;(1-\frac{x}{s})u] du ds\ .$$

Aplicando (11) y simplificando se obtiene,

$$J_x^{\alpha-\gamma,\beta-\delta-\mu,\gamma+\eta,\mu}\, J_x^{\gamma,\delta,\eta,\mu} f = \frac{x^{-\mu}}{\Gamma(\alpha)} \int_x^\infty s^{-\alpha-\beta}(s-x)^{\alpha-1}\ .$$

$$.\ F(\alpha+\beta+\mu,-\eta;\alpha\,; 1-\frac{x}{s})\, f(s) ds = J_x^{\alpha,\beta,\eta,\mu} f.$$

De manera análoga, se demuestra que:

$$J_x^{\alpha,\beta,\eta,\mu} f = J_x^{\gamma,\delta-\mu,\eta-\beta+\delta-\mu,\mu}\, J_x^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta,\mu} f, \qquad (35)$$
$$\alpha > \gamma > 0,\ \beta > \delta > 0\ .$$

Aplicando (34) y (35) se comprueba que

$$J_x^{\gamma,\delta,\alpha+\eta,\mu}\, J_x^{\alpha,\beta,\eta,\mu} f = J_x^{\alpha+\gamma,\beta+\delta+\mu,\eta,\mu} f, \qquad (36)$$

$$J_x^{\gamma,\delta,\eta-\beta-\gamma-\delta,\mu}\, J_x^{\alpha,\beta,\eta,\mu} f = J_x^{\alpha+\gamma,\beta+\delta+\mu,\eta-\gamma-\delta+\mu,\mu} f, \qquad (37)$$

Otras Fórmulas para $J_x^{\alpha,\beta,\eta,\mu} f$

Seguidamente se presentan otras fórmulas que involucran la transformada de Mellin y el Operador $J_x^{\alpha,\beta,\eta,\mu} f$

Si en (18), (19), (20) y (21), sustituimos λ por $-\lambda+\alpha+\gamma$, se tiene:

$$I(\gamma,\delta,\zeta,\mu;\alpha,\beta,\eta,\mu; W) =$$
$$= I(-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\zeta,\mu;\lambda,-\lambda+\alpha+\beta,-\lambda+\alpha+\eta,\mu; W)\ , \qquad (38)$$

$$= I(-\lambda+\alpha+\gamma,\lambda+\beta+\eta-\gamma+\delta-\zeta+\mu,\zeta,\mu;\lambda,-\lambda-\eta+\gamma+\zeta-\mu,$$
$$-\lambda-\beta+\gamma+\zeta-\mu,\mu; W)\ , \qquad (39)$$

$$= I(-\lambda+\alpha+\gamma,\lambda-\alpha+\zeta-\eta-\mu,\eta+\delta+\mu,\mu;\lambda,-\lambda+\alpha+\beta+\eta+\delta-\zeta+\mu,$$
$$-\lambda+\alpha+\eta,\mu; W)\ , \qquad (40)$$

$$= I(-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\eta+\delta+\mu,\mu;\lambda,-\lambda+\gamma+\delta,-\lambda-\beta+\gamma+\zeta-\mu,\mu; W) \qquad (41)$$

Aplicando (38), (39), (40), (41) y siguiendo un procedimiento análogo al usado en la demostración de (22) se establecen:

$$J_x^{\alpha,\beta,\eta,\mu}\, J_x^{\gamma,\delta,\zeta,\mu} f = J_x^{\lambda,-\lambda+\alpha+\beta,-\lambda+\alpha+\eta,\mu}\ .$$
$$J_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\zeta,\mu} f, \qquad (42)$$

$$J_x^{\alpha,\beta,\eta,\mu}\, J_x^{\gamma,\delta,\zeta,\mu} f = J_x^{\lambda,-\lambda-\eta+\gamma+\zeta-\mu,-\lambda-\beta+\gamma+\zeta-\mu,\mu}\ .$$
$$.\ J_x^{-\lambda+\alpha+\gamma,\lambda+\beta+\eta-\gamma+\delta-\zeta+\mu,\zeta,\mu} f \qquad (43)$$

$$J_x^{\alpha,\beta,\eta,\mu}\, J_x^{\gamma,\delta,\zeta,\mu} f = J_x^{\lambda,-\lambda+\alpha+\beta+\eta+\delta-\zeta+\mu,-\lambda+\alpha+\eta,\mu} \cdot$$
$$\cdot\, J_x^{-\lambda+\alpha+\gamma,\lambda-\alpha+\zeta-\eta-\mu,\eta+\delta+\mu,\mu} f, \quad (44)$$

$$J_x^{\alpha,\beta,\eta,\mu}\, J_x^{\gamma,\delta,\zeta,\mu} f = J_x^{\lambda,-\lambda+\gamma+\delta,-\lambda-\beta+\gamma+\zeta-\mu,\mu} \cdot$$
$$\cdot\, J_x^{-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\eta+\delta+\mu,\mu} f. \quad (45)$$

Si $\mu = 0$ se obtiene como casos particulares resultados dados por Saigo [5].

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