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Isotopic lifting on differential geometries

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Abstract

In the first paper of this series we have introduced the isotopies of the differential calculus and of Newton's equations of motion. In the second paper we used these results to construct the isotopies of analytic and quantum mechanics. In this third paper we apply the preceding results for the construction of the isotopies of conventional differential geometries, such as the symplectic and Riemannian geometries. The primary motivation is that, in their conventional formulation, these geometries are local-differential. As such, the are only valid for the exterior dynamical problem of point-like test bodies moving in the homogeneous and isotropic vacuum. The isotopic geometries result instead to be valid for the interior dynamical problem of extended and deformable test bodies moving within inhomogeneous and anisotropic physical media with conventional local-differential and variationally self-adjoint as well as nonlocal-integral and variationally nonselfadjoint resistive forces. In this paper we show that the isotopic geometries preserve all original axioms to such and extent that they coincide at the abstract level with the conventional geometries.

Key words: Isotopies, isosymplectic geometry, isoriemannian geometry.

Levantamiento isotópico de geometrías diferenciales

Resumen

En el primer trabajo de esta serie presentamos las isotopías del cálculo diferencial y de las ecuaciones de movimiento de Newton. En el segundo, utilizamos estos resultados para construir las isotopías de la mecánica cuántica y analítica. En este tercer trabajo aplicamos los resultados anteriores para la construcción de las isotopías de geometrías diferenciales convencionales, tales como las geometrías simpléticas y de Riemann. La motivación primaria es que, en su formulación convencional, estas geometrías son locales-diferenciales. Como tales, sólo son válidas para el problema dinámico externo de cuerpos de prueba puntuales que se mueven en un vacío homogéneo e isotópico. En cambio, estas geometrías isotópicas resultan ser válidas para el problema dinámico interno de cuerpos experimentales extendidos y deformables moviéndose en medios físicos no homogéneos y anisotópicos con fuerzas resistivas tanto locales diferenciales y variablemente auto-lindantes como no locales-integrales y variablemente no auto-lindantes. En este trabajo demostramos que las geometrías isotópicas preservan todos los axiomas originales hasta un punto tal que coinciden, a nivel abstracto, con las geometrías convencionales.

Palabras claves: Isotopías, geometría isosimplética, geometría isoriemaniana.

1. Statement of the problem.

The contemporary geometries, such as the *symplectic geometry* (see, e.g., [1] or [23] for a review and comprehensive literature) and the *Riemannian geometry* [14] (see, e.g., [1] for historical profiles and [12] for a recent account in local coordinates) have permitted during this century truly outstanding achievement for a deeper understanding of the physical structure of the Universe (see Einstein's collected papers [29]).

Nevertheless, these geometries are *local-differential* and, as such, they have well defined limitations in their applications to physical systems which are expressed by the historical distinction between

1) the *exterior dynamical problems*, consisting of point-like test bodies moving in the homogeneous and isotropic vacuum, such as a space-ship in a stationary orbit in vacuum around Earth or a proton in a particle accelerator; and

2) the *interior dynamical problems*, consisting of extended and deformable test bodies moving within inhomogeneous and anisotropic physical media, such as a space-ship during re-entry in our atmosphere, or a proton in the core of a collapsing star.

This distinction was introduced by Lagrange [10], Hamilton [7] and other founders of analytic dynamics. In the preceding paper [25] (hereinafter referred to as Paper I) we have recalled that Newton's equations of motion contain local-differential terms describing action-at-a-distance, potential forces and representable via a first-order Lagrangian, plus nonlocal-integral and nonlagrangian terms representing precisely the resistive forces of interior dynamical problems. As recalled in paper [26] (hereinafter referred to as Paper II), Lagrange and Hamilton formulated their celebrated equations, not in the form of widespread use in the contemporary mathematical and physical literature, but rather in the form with *external terms* representing precisely the additional forces of interior conditions.

The distinction between exterior and interior dynamical problems was fully adopted during the early studies in gravitation [29], as illustrated, e.g., by Schwartzschild's *two* papers, the first famous paper [27] on the *exterior* gravitational problem and the second little known paper [28] on the *interior* problem. The same distinction was also kept in the early well written treatises in gravitation (see, e.g., the monograph by Bergmann [2] with a preface by Einstein).

Particularly significative is the adoption in these early studies of the Riemannian geometry and ensuing physical theories as being *exactly* valid for the exterior problem and *approximately* valid for the interior problem [28].

Regrettably, the above distinction was progressively relaxed during the second part of this century, up to the current condition of virtual complete silence in the specialized mathematical and physical literature.

In particular, the distinction was eliminated where it is needed most, in the interior conditions of gravitational collapse, black holes, big bang and all that, where the interior dynamical problem reaches its extreme conditions. In fact, collapsing stars are not a collection of ideal point-particles (as *necessary* for the applicability of the symplectic and Riemannian geometries), but in the physical reality they are composed of extended and hyperdense protons and neutrons in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. These conditions imply the most general conceivable interior field equations which are arbitrarily nonlinear in the *velocities and accelerations*, as well as nonlocal integral and nonlagrangian. The lack of exact validity of conventional local-differential geometries under the latter conditions is then beyond scientific doubts.

The elimination of interior dynamical conditions from the contemporary mathematical and physical literature has been essentially done via the reduction of interior problems to a collection of exterior ones in vacuum. For instance, a space-ship during re-entry with nonlocal-integral and variationally non-self-adjoint forces is reduced to an ideal collection of point-like elementary particles. The expectation is that, in this reduction, conventional geometries are re-established at the particle level.

By keeping in kind that a quantum version of gravity (which is a pre-requisite for the reduction) has not yet been achieved in a form acceptable by the scientific community at large, recent studies have established that the above reduction is *mathematically and physically impossible*. In fact, there exist nowadays the so-called *No-Reduction Theorems* [24] which establish that a space-ship during re-entry *on a decaying orbit with a monotonically decaying angular momentum* cannot be consistently reduced to a finite collection of point-like particles *on stable orbits with conserved angular momenta* (as necessary for the exact applicability of conventional geometries and symmetries). Additional reasons for the lack of exact applicability of conventional geometries for interior conditions are studied in ref. [24].

At any rate, one of the pillars of the Riemannian geometry is the representation of the *homogeneity and isotropy of the vacuum*. As such, the same geometry cannot effectively represent the *inhomogeneity and anisotropy of physical media* such as our atmosphere.

Also, it is known in the literature (see, e.g., E. Cartan [4]) that the Riemannian geometry can recovers only *some* but not all Newtonian systems of our physical reality. A typical examples is given by missiles in atmosphere which nowadays have reached such speeds to require drag forces up to the seventh power in the velocity and more,

$$F^{NSA} = -\sum_{k=1,2,3,4,5,6,7} \gamma_k \dot{x}^k , \qquad (1.1)$$

where the γ's are positive-definite constants. Force (1.1) is evidently a truncated power series *approximation* of the actual nonlocal-integral forces depending on the shape of the missile. The inapplicability of the Riemannian geometry for interior systems with forces (1.1) is then beyond scientific doubts.

The fundamental geometric problem addressed in this paper is therefore *the identification of novel geometries specifically constructed for interior dynamical problems, that is, capable of representing extended and deformable test bodies mowing within inhomogeneous and anisotropic physical media with arbitrarily nonlinear, nonlocal-integral and nonlagrangian forces.* Moreover, to be effective for physical applications (particularly for experimental verifications), *the new geometries must admit the original geometries as a particular case (i.e., be covering geometries) and permit a clear and unambiguous separation between the exterior and interior contributions*.

Without any claim of uniqueness, this author selected the isotopic methods for the construction of the new geometries, as originally presented in ref.s [18,19,20] and then studied in detail in [24,25] under the names of *isoeuclidean, isominkowskian, isosymplectic, isoaffine* and *isoriemannian geometries*, generically referred to as *isogeometries*. The selection was done on purely *physical* grounds because the isotopies permit the preservation of the original geometric axioms, thus preserving the Einsteinian axioms as well. Other geometries, such as conventional integral geometries, do not generally preserve the original axioms, thus creating the problem of identifying new physical axioms and, after that, of proving them experimentally.

Moreover, the isogeometries admit conventional geometries as a particular case and clearly separate exterior and interior contributions. In fact, the isotopies of conventional space-time geometries are based on the lifting of the conventional (3+1)-dimensional unit $I = \mathrm{diag.}(1, 1, 1, 1)$ into the most general possible (3+1)-dimensional isounits of Kadeisvili Class I [8] (sufficiently smooth, bounded, nowhere singular, real valued, symmetric and positive-definite) with nonlinear and nonlocal-integral dependence on coordinates x, their derivatives $\dot{x}$, $\ddot{x}$, ..., with respect to an independent variable and any needed additional variable. In their diagonal form, the isounits can be written

$$\hat{1} = \mathrm{diag.}(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2})\,\Gamma(x, \dot{x}, \ddot{x}, ...), \qquad (1.2)$$

where $\mathrm{diag.}(n_1^{-2}, n_2^{-2}, n_3^{-2})$ represents the nonspherical-deformable shape of the test body considered, n_4^{-2} geometrizes its density, and $\Gamma(x, \dot{x}, \ddot{x}, ...)$ represents the nonlinear, nonlocal and nonhamiltonian interactions (see [24] for details, applications and verifications). Conventional action-at-a-distance interactions are represented via the conventional potential.

In this way, the isogeometries recover identically and unambiguously the conventional geometries of the exterior problem in vacuum for $\hat{1} \equiv I$ and permit a clear separation suitable for experimental verifications between exterior and interior contributions via the deviation of $\hat{1}$ from I.

The first isogeometries were constructed by this author [18,19,20] with *isotopies based on the degree of freedom of the conventional multiplication.* The main characteristic of these isotopies is that the original geometric axioms are preserved on isospaces over isofield, as well as on their projection into the original space over ordinary fields.

In this paper we introduce, apparently for the first time, *isogeometries constructed via the isodifferential calculus.* As we shall see, the latter are more general than the former because the original geometric axioms are indeed preserved in isospaces over isofields, but not necessarily in their projection in the original spaces over ordinary fields.

The *isoeuclidean geometry* has been studied in detail in monograph [23]. Its reformulation in term of the isodifferential calculus is elementary and implies no major structural change. The *isominkowskian geometry* can be obtained as the tangent geometry to the isoriemannian one. We shall therefore limit ourselves to the study of the *isosymplectic and isoriemannian geometries based on the isodifferential calculus.*

Our analysis is mainly local, owing to the need to identify geometries which are specifically applicable in the given inertial frame of the observer (see Paper II). Abstract, coordinate-free treatments are therefore merely indicated. All results of this paper can be easily extended to isounits of Kadeisvili Class II (same property of Class I except that $\hat{1}$ is negative-definite) and of Class III (union of Class I and II). However the extension to Classes IV (Class III plus singular isounits) and V (Class IV plus arbitrary isounits, including discontinuous isounits) requires specific studies.

The reader should be aware that the isogeometries of Class II, called *isodual isogeometries,* have resulted to be particularly suited for a novel treatment of *antimatter* [24]. In fact, the operator formulation of the antiautomorphic map

$$\hat{1} > 0 \quad \rightarrow \quad \hat{1}^d = -\hat{1} < 0, \tag{1.3}$$

called by this author *isoduality,* is equivalent to charge conjugation. This has allowed the initiation of: novel studies, such as the first astrophysical studies beginning at the *classical* level and then persists under isoquantization (Paper II) of stars, galaxies and quasars as made-up entirely of antimatter; experimentally verifiable studies of antigravity; and others [24]. By comparison, conventional methods permit the treatment of antimatter only at the level of *second quantization,* as well known.

The isogeometries of Class III have stimulated a new *cosmology* in which the Universe can be made up of equal distributions of matter and antimatter with intriguing features, such as null total energy, null total time, etc. [24] (because isodual isofields [22], having a negative-definite norm, imply physical characteristics of antimatter opposite to those of matter, resulting in null total characteristics for equal distribution of matter and antimatter).

Finally, the reader should be aware that the isogeometries of Kadeisvili Class IV have resulted to be particularly significant for further advances on *gravitational singularities* for both matter and antimatter. In fact, conventional (3+1)-dimensional Riemannian metrics g(x) always admit the factorization of the Minkowskian metric,

$$g(x) = \hat{T}_{gr}(x)\,\eta, \qquad \eta = \mathrm{diag.}\,(1, 1, 1, -1). \tag{1.4}$$

Gravitational horizons (singularities) are then given by the zeros of the isotopic element $\hat{T}_g$ (isounit $\hat{1}_{gt}$)

Gravitational horizons: $T_{gr}(x) = 0$; Gravitational singularities:

$$\hat{1}_{gr}(x) = 0. \tag{1.5}$$

But the above representation has no effective contribution from the internal nonlinear, nonlocal and nonlagrangian effects. The isoriemannian geometry therefore permits the enlargement of the definition of gravitational horizons and singularities for the inclusion of interior nonlinear and nonlocal effects which is achieved via the zeros of the general isotopic elements and isounits, respectively,

$$\text{Gravitational horizons: } T_{gr}(x, \dot{x}, \ddot{x}, \ldots) = 0; \tag{1.5a}$$

$$\text{Gravitational singularities: } \hat{1}_{gr}(x, \dot{x}, \ddot{x}, \ldots) = 0. \tag{1.5b}$$

The latter comments have been made in the hope of stimulating mathematical studies on the isotopies of

Kadeisvili Class IV which are vastly unknown at this writing.

One should also note that the isotopies have permitted the construction of the universal symmetry for all possible (3+1)-dimensional, exterior and interior gravitation, called isopoincare' symmetry $\hat{P}(3.1)$, and that symmetry has resulted to be locally isomorphic to the Poincare' symmetry P(3.1) [24] (see independent review [9]). The isosymmetry is again constructed via the Minkowskian factorization of any given exterior or interior metric $g = \hat{T}_{gr}\eta$, and the reconstruction of the conventional Poincare' symmetry with respect to the isounit $\hat{1}_{gr} = \hat{T}_{gr}^{-1}$. The local isomorphism $\hat{P}(3.1) \approx P(3.1)$ follows from the positive-definiteness of the isotopic element $\hat{T}_{gr}$ for all physical models of gravitation (outside the gravitational horizon). The universality of the $\hat{P}(3.1)$ isosymmetry for all infinitely possible gravitations follows from the unrestricted functional dependence of the isounit $\hat{1}_{gr}(x, \dot{x}, \ddot{x}, ...)$.

In turn, the achievement of a universal symmetry for gravitation has stimulated numerous novel studies, such as a possible unification of relativistic quantum mechanics and gravitation via the embedding of gravitation in the *unit* of conventional theories, and others [24].

Finally, the reader should remember from the introduction of Paper I that the isogeometries are a particular case of the *genogeometries* 16,23] (in which case the totally symmetric character of the genometric is relaxed) and that, in turn the genogeometries are a particular case of the *multivalued hypergeometries* (in which the unit can assume an ordered set of values).

2. Isosymplectic geometry.

We identify in this section the isotopies of the symplectic geometry, called *isosymplectic geometry* for short, as the geometry underlying the isohamilton equations and related Lie-isotopic theory of Paper II. These isotopies were first studied by this author in ref. [19], then subjected to deeper studies in subsequent papers and in monograph [23] via the lifting of the units and of the conventional associative product. The formulation of the isosymplectic geometry based on the isodifferential calculus is presented here for the first time.

Unless otherwise stated, all quantities are assumed to satisfy the needed continuity conditions, e.g., of being of class $\hat{C}^{\infty}$ and all neighborhoods of a point are assumed to be star-shaped or have an equivalent topology. For a presentation of the conventional symplectic geometry we refer to [1], while comprehensive literature in the field is available in ref. [16] and it is omitted here for brevity.

Let $\hat{M}(\hat{E}) = \hat{M}(\hat{E}(,\hat{\delta},\hat{R})$ be an N-dimensional Tsagas-Sourlas isomanifold [20,31] modified according to Definition 3 of Paper I on the isoeuclidean space $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isoreals $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ with N×N-dimensional isounit $\hat{1} = (\hat{1}^i{}_j)$, i, j = 1, 2, ..., N, of Kadeisvili [8] Class I and local chart $\hat{x} = \{\hat{x}^k\}$. A *tangent isovector* $\hat{X}(\hat{m})$ at a point $\hat{m} \in M(\hat{E})$ is an isofunction defined in the neighborhood $\hat{N}(\hat{m})$ of $\hat{m}$ with values in $\hat{R}$ satisfying the *isolinearity conditions*

$$\hat{X}_m(\hat{a} \hat{\times} \hat{f} + \beta \hat{\times} g) = \hat{a} \hat{\times} \hat{X}_{\hat{m}}(\hat{f}) + \beta \hat{\times} \hat{X}_{\hat{m}}(\hat{g}), \quad \hat{X}_{\hat{m}}(\hat{f} \hat{\times} \hat{g}) =$$

$$= \hat{f}(\hat{m}) \hat{\times} \hat{X}_{\hat{m}}(\hat{g}) + \hat{g}(\hat{m}) \hat{\times} \hat{X}_m(\hat{f}), \qquad (2.1)$$

for all $\hat{f}$, $\hat{g} \in \hat{M}(\hat{E})$ and $\hat{a}$, $\hat{\beta} \in \hat{R}$, where $\hat{\times}$ is the isomultiplication in $\hat{R}$ and the use of the symbol ^ means that the quantities are defined on isospaces.

The collection of all tangent isovectors at $\hat{m}$ is called the *tangent isospace* and denoted TM($\hat{E}$). The *tangent isobundle* is the 2N-dimensional union of all possible tangent isospaces when equipped with an isotopic structure (see below). The *cotangent isobundle* $T^*\hat{M}(\hat{E})$ is the dual of the tangent isobundle and it is defined with respect to the isounit $\hat{1}_2 = \text{diag.}(\hat{1}, \hat{T}) = \text{diag.}(\hat{T}^{-1}, \hat{1}^{-1})$, with the understanding pointed our in the preceding section that more general isounits of the type $\hat{1}_2 = \text{diag.}(\hat{1}, \hat{W}^{-1})$, $\hat{W} \neq \hat{1}$, are possible because of the independence of $\hat{x}$ and $\hat{p}$.

Let $\hat{b} = \{\hat{b}^{\mu}\} = \{\hat{x}^k, \hat{p}_k\}$, μ = 1, 2, ..., 2N, be a local chart of $T^*M(\hat{E})$. An *isobasis* of $T^*M(\hat{E})$ is, up to equivalence, the (ordered) set of isoderivatives $\hat{\partial} = \{\hat{\partial}/\hat{\partial}\hat{b}^{\mu}\} = \{\hat{T}_{2\mu}{}^{\nu}\partial/\partial b^{\nu}\}$. A generic elements $\hat{X} \in T^*M(\hat{E})$ can then be written $\hat{X} = \hat{X}^{\mu}(\hat{m})\, \hat{\partial}/\hat{\partial}\hat{b}^{\mu}$.

The *fundamental one-isoform* on $T^*\hat{M}(\hat{E})$ is given in the local chart $\hat{b}$ by

$$\hat{\theta} = \hat{R}^{\circ}_{\mu}(\hat{b})\,\hat{\partial}\hat{b}^{\mu} = \hat{R}^{\circ}_{\mu}(b)\,1_{2}{}^{\mu}{}_{\nu}\,d\hat{b}^{\nu} = \hat{p}_k\,\hat{\partial}\,\hat{x}^k = \hat{p}_k\,1^k{}_i\,d\hat{x}^i,$$

$$\hat{R}^{\circ} = \{\,\hat{p}, \hat{0}\,\}. \tag{2.2}$$

The space $T^*\hat{M}(\hat{E})$, when equipped with the above one-form, is an *isobundle* denoted $T_1{}^*\hat{M}(\hat{E})$. The isoexact, nowhere degenerate, *isosymplectic two-isoform* in isocanonical realization is given by

$$\hat{\omega} = \hat{\partial}\,\hat{\theta} = \hat{\partial}\,(\hat{R}^{\circ}{}_{\mu}\,\hat{\partial}\hat{b}^{\mu}) = \omega_{\mu\nu}\,\hat{\partial}\hat{b}^{\mu} \wedge \hat{\partial}\hat{b}^{\nu} =$$

$$= 2\,\hat{\partial}\hat{x}^k \wedge \hat{\partial}\hat{p}_k = 1^k{}_i\,d\hat{x}^i \wedge T_k{}^j\,d\hat{p}_j \equiv d\hat{x}^k \wedge d\hat{p}_k \equiv \omega \tag{2.3}$$

The isospace $T^*\hat{M}(\hat{E})$, when equipped with the above two-isoform, is an *isosymplectic isomanifold* in isocanonical realization denoted $T_2{}^*\hat{M}(\hat{E})$. The *isosymplectic geometry* is the geometry of the isosymplectic isomanifolds.

The last identity in (2.3) show that *the isosymplectic isocanonical two-isoform $\hat{\omega}$ formally coincides with the conventional symplectic canonical two-form ω*. The abstract identity of the symplectic and isosymplectic geometries is then evident. This illustrates on geometric grounds Bruck's [3] statement to the effect that "the isotopies are so natural to keep in un-noticed". However, one should remember that the underlying metric is isotopic, that $\hat{p}_k = T_k{}^i p_i$, where p_i is the variable of the conventional canonical realization of the symplectic geometry. and that identity (2.3) no longer holds for the more general isounits $1_2 = \text{diag.}\,(1, W^{-1})$. Also, the symplectic geometry is local-differential, while the isosymplectic geometry admits nonlocal-integral terms when embedded in the isounit.

A vector isofield $\hat{X}(\hat{m})$ defined on the neighborhood $N(\hat{m})$ of a point $\hat{m} \in T_2{}^*\hat{M}(\hat{E})$ with local coordinates $\hat{b}$ is called *isohamiltonian* when there exists an isofunction $\hat{H}$ on $\hat{N}(\hat{m})$ over $\hat{R}$ such that $\hat{X} \lrcorner\, \hat{\omega} = -\,\hat{\partial}\hat{H}$, i.e.,

$$\omega_{\mu\nu}\,\hat{X}^{\nu}(\hat{m})\,\hat{\partial}\hat{b}^{\mu} = \hat{\partial}\hat{H}(\hat{m}) = (\,\hat{\partial}\hat{H} \,/\, \hat{\partial}\hat{b}^{\mu})\,\hat{\partial}\hat{b}^{\mu}, \tag{2.4}$$

which are equivalent to isohamilton equations (II.4.10a). The isosymplectic geometry is therefore the geometry underlying the isohamiltonian mechanics of Paper II.

It is straighforward to construct isoforms $\hat{\Phi}_p$ of arbitrary order p. The proof of the following property then follows from the properties of the isodifferential calculus.

Lemma 1 (Isopoincare' Lemma)

Under the assumed smoothness and regularity conditions, isoexact p-isoforms are closed, i.e.,

$$\hat{\partial}\,\hat{\Phi}_p = \hat{\partial}\,(\,\hat{\partial}\hat{\Phi}_{p-1}\,) \equiv 0. \tag{2.5}$$

For the two-dimensional case (see, e.g., [15] or [17]), the conventional Poincaré lemma is known to provide the necessary and sufficient conditions in geometric form for the contravariant tensor $\omega^{\mu\nu} = [(\omega_{\alpha\beta})^{-1}]^{\mu\nu}$ to be Lie, i.e., for brackets (II.3.9) to satisfy the Lie algebra axioms, where $\omega_{\mu\nu}$ is the canonical symplectic tensor. In this way, the symplectic geometry is the geometry underlying Lie's theory.

The isopoincaré lemma for the two-dimensional case provides the necessary and sufficient conditions for the same contravariant tensor $\omega^{\mu\nu}$ to be , this time, Lie-isotopic, i.e., for the isobrackets (II.3.21) to verify the Lie axioms in isospaces over isofields [23]. The isosymplectic geometry is therefore the geometry underlying the Lie-Santilli isotheory.

The *general one-isoform* in the local chart $\hat{b}$ is given by

$$\hat{\theta} = \hat{R}_{\mu}(\hat{b})\,\hat{\partial}\hat{b}^{\mu} = \hat{R}_{\mu}(\hat{b})\,1_2{}^{\mu}{}_{\nu}(t, \hat{b}, \hat{\partial}\hat{b}/\hat{\partial}\hat{t}, \ldots)\,d\hat{b}^{\nu},\ \hat{R} =$$

$$= \{\,\hat{P}(\hat{x},\hat{p}),\ \hat{Q}(\hat{x}, \hat{p})\,\}. \tag{2.6}$$

The *general isosymplectic isoexact two-isoform* in the same chart is then given by

$$\hat{\Omega} = \hat{\partial}\,(\,\hat{R}_{\mu}(\hat{b})\,\hat{\partial}\hat{b}^{\nu}\,) = \hat{\Omega}_{\mu\nu}\,(\hat{t}, \hat{b}, \hat{\partial}\hat{b}/\hat{\partial}\hat{t}, \ldots)\,\hat{\partial}\hat{b}^{\mu} \wedge \hat{\partial}\hat{b}^{\nu}, \tag{2.7a}$$

$$\hat{\Omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_{\nu}}{\hat{\partial}\hat{b}^{\mu}} - \frac{\hat{\partial}\hat{R}_{\mu}}{\hat{\partial}\hat{b}^{\nu}} = T_{2\mu}{}^{\alpha}\,\frac{\hat{\partial}\hat{R}_{\nu}}{\hat{\partial}\hat{b}^{\alpha}} - T_{2\nu}{}^{\alpha}\,\frac{\hat{\partial}\hat{R}_{\mu}}{\hat{\partial}\hat{b}^{\alpha}}. \tag{2.7b}$$

One can see that, while at the canonical level the exact the two-form ω and its isotopic extension $\hat{\omega}$ formally coincide, *this is no longer the case for exact, but* *arbitrary two forms* Ω and $\hat{\Omega}$ in the same local chart.

Note that the isoform $\hat{\Omega}$ is isoexact, $\hat{\Omega} = \hat{\partial}\hat{\Theta}$, and therefore isoclosed, $\hat{\partial}\hat{\Omega} \equiv 0$, in isospace over the isofield $\hat{R}$. However, if the same isoform $\hat{\Omega}$ is projected in ordinary space and called Ω, it is no longer necessarily exact and, therefore, it is not generally closed, $d\Omega \neq 0$. These properties prove the following

Lemma 2 (General Lie-Santilli Brackets)

Let $\hat{\Omega} = \hat{\Omega}_{\mu\nu}\, \hat{\partial}\hat{b}^{\mu} \wedge \hat{\partial}\hat{b}^{\nu}$ *be a general exact two-isoform,* $\hat{\Omega} = \hat{\partial}\hat{\Theta} = \hat{\partial}(\hat{R}_{\mu}\hat{\partial}\hat{b}^{\mu})$. *Then the brackets among sufficiently smooth and regular isofunctions* $\hat{A}(\hat{b})$ *and* $\hat{B}(\hat{b})$ *on* $\hat{T}_2{}^*M(\hat{E})$

$$[\hat{A}, \hat{B}]_{isot.} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^{\mu}}\, \hat{\Omega}^{\mu\nu}\, \frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^{\nu}}, \qquad (2.8a)$$

$$\hat{\Omega}^{\mu\nu} = \left[\left(\frac{\hat{\partial}\hat{R}_{\alpha}}{\hat{\partial}\hat{b}^{\beta}} - \frac{\hat{\partial}\hat{R}_{\beta}}{\hat{\partial}\hat{b}^{\alpha}} \right)^{-1} \right]^{\mu\nu}, \qquad (2.8b)$$

satisfy the Lie-Santilli axioms in isospace (but not necessarily the same axioms when projected in ordinary spaces).

An important property of the symplectic geometry is Darboux's Theorem [5] which expresses the capability of reducing arbitrary symplectic two-forms to the canonical form or, equivalently, the reduction of Birkhoff's to Hamilton's equations. (Paper II) The following additional property completes the axiom-preserving character of the isotopies of the symplectic geometry.

Theorem 1 (Isodarboux Theorem)

2N-dimensional isocotangent bundle $\hat{T}_2{}^*M(\hat{E})$ *equipped with a nowhere degenerate, exact,* $\hat{C}^{\infty}$ *two-isoform* $\hat{\Omega}$ *in the local chart* $\hat{b}$ *is an isosymplectic manifold if and only if there exists coordinate transformations* $\hat{b} \to \hat{b}'(\hat{b})$ *under which* $\hat{\Omega}$ *reduces to the isocanonical two-isoform* $\hat{\omega}$, *i.e.*

$$\frac{\hat{\partial}\hat{b}^{\mu}}{\hat{\partial}\hat{b}'^{\alpha}}\, \hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}'))\, \frac{\hat{\partial}\hat{b}^{\nu}}{\hat{\partial}\hat{b}'^{\beta}} = \omega_{\alpha\beta}. \qquad (2.9)$$

PROOF. Suppose that the transformation $\hat{b} \to \hat{b}'(b)$ occurs via the following intermediate transform $\hat{b} \to \hat{b}''(\hat{b}) \to \hat{b}'(b''(b))$. Then there always exists a transform $\hat{b} \to \hat{b}''$ such that

$$(\hat{\partial}\hat{b}^{\rho} / \hat{\partial}\hat{b}''^{\sigma})(\hat{b}'') = \hat{1}^{\rho}{}_{\sigma}(\hat{b}(\hat{b}'')), \qquad (2.10)$$

under which the general isosymplectic tensor $\hat{\Omega}_{\mu\nu}$ reduces to the Birkhoffian form when recompute in the $\hat{b}$ chart

$$\frac{\hat{\partial}\hat{b}^{\mu}}{\hat{b}''^{\alpha}}\, \hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}''))\, \frac{\hat{\partial}\hat{b}^{\nu}}{\hat{\partial}\hat{b}''^{\beta}}\Big|_{\hat{b}''} = \left(\frac{\hat{\partial}\hat{R}_{\nu}}{\hat{\partial}\hat{b}^{\alpha}} - \frac{\hat{\partial}\hat{R}_{\mu}}{\hat{\partial}\hat{b}^{\nu}} \right)\Big|_{\hat{b}''} = \Omega_{\alpha\beta}\,\Big|_{\hat{b}''}. \qquad (2.11)$$

The existence of a second transform $\hat{b}'' \to \hat{b}'$ reducing $\Omega_{\alpha\beta}$ to $\omega_{\alpha\beta}$ is then known to exist (see, e.g., [1,15,17]). This proves the necessity of the isodarboux chart. The sufficiency is proved as in the conventional case. Q.E.D.

The isotopies of the remaining aspects of the symplectic geometry (Lie derivative, global treatment, etc.) can be constructed along the preceding lines and are omitted for brevity.

Remark 1. The symplectic geometry in canonical realization can geometrize in the given b-chart only a subclass of Newtonian systems, namely, conservative systems plus a restricted class of nonconservative systems called *nonessentially nonselfadjoint* [15]. The remaining systems can only be geometrized via their representation with respect to an arbitrary symplectic two-form and its reduction to the canonical form via the Darboux's transforms. However, the Darboux transforms are nonlinear and therefore, as recalled in Paper II, they cannot be realized in laboratory and imply the loss of conventional relativities because of the loss of the inertial character of the original frame.

Remark 2. The *direct universality* of the *conventional* symplectic geometry for the characterization of all possible local, analytic and regular Newtonian systems (universality) in the frame of the experimenter (direct universality), was proved in ref. [17] via the use of the general one-forms on the ordinary cotangent bundle $T^*M(E) = T^*M[E(x,\delta,R)]$ in the local realization

$$\Theta = R_{\mu}(b)\, db^{\mu}, \qquad (2.12)$$

with corresponding general, exact, symplectic two-form

$$\Omega = \Omega(b)_{\mu\nu}\, db^{\mu} \wedge db^{\nu}, \quad (2.13)$$

where $\Omega_{\mu\nu}$ is the Birkhoffian tensor (II.1.11). A vector field X(m) in the neighborhood of a point $m \in T^*M(E)$ which is not Hamiltonian in the given chart b results to be always Birkhoffian in the same chart, i.e., when a function H on N(m) such that X_]ω = -dH does not exist in the b-chart, there always exist a Birkhoffian tensor $\Omega_{\mu\nu}(b)$ such that X_]Ω = -dH. The maps within a fixed b-chart $\theta \rightarrow \Theta$ and $\omega \rightarrow \Omega$ were identified in ref. [25] as a first form of isotopies of the symplectic geometry in canonical realization.

Remark 3. Despite the achievement of the above direct universality, the symplectic geometry continues to be insufficient for recent applications owing to its local-differential character.

A second isotopy of the symplectic geometry for the characterization of nonlocal, integral terms was submitted by this author [18] via the lifting of the unit and of the associative product while preserving the conventional differential calculus. For instance, the isocanonical one-form on $T^*\hat{M}(\hat{E})$ in the above formulation is given by

$$\hat{\theta} = \hat{R}^{\circ}_{\mu}(\hat{b})\, \hat{T}_{2}{}^{\mu}{}_{\nu}\, d\hat{b}^{\nu}. \quad (2.14)$$

and, as such, it *coincides* with one-isoform (2.2) except for the replacement of the isounit with the isotopic element. The isotopic degrees of freedom of the product of the former are then transferred to those of the differentials in the latter. However, two-isoforms result to be different in the two approaches, as one can verify (see ref. [23], Sect. 5.4 for brevity).

The above second isotopy of the symplectic geometry preserves all conventional axioms, including the Poincaré Lemma, the Darboux's Theorem, etc. Also, the latter theorems hold in both isospaces as well as in their projection into the conventional spaces. In particular, the generalized brackets were Lie-isotopic in both isospace and in their projection in the conventional space.

The drawback of the above isotopy is that it implies the loss of the basic unit $\hat{1}_2$ in the transition from one- to two-isoforms evidently due to the use of the conventional calculus (see also ref.s [23] Sect. 5.4 for brevity). In turn, the lack of invariance of the unit has serious problematic aspects of physical character, e.g., in the conduction of measurements.

In this section we have introduced the third isotopy of the symplectic geometry studied by this author, this time based on the isodifferential calculus. Its main advantages over the preceding isotopies is its remarkable simplicity, as well as the preservation of the basic unit $\hat{1}_2$ = diag. $(\hat{1}, \hat{T})$ for isoforms of arbitrary order, thus permitting its consistent application for measurements. Another advantage is that *the conventional coordinate-free treatment of the symplectic geometry can be preserved in its entirety for the characterization of the isosymplectic geometry submitted in this section and merely subjected to a more general realization of the symbols such as* dx, dH, *etc.* In different terms, the contemporary coordinate-free formulation of the symplectic geometry(as available, e.g., in [1]) can be left completely unchanged for the characterization of the covering isosymplectic geometry, and merely subject the isodifferentials to a more general realization.

Remark 4. The isosymplectic geometry of this section is particularly suited for the isotopies of symplectic quantization first studied by Lin [11] and then treated in [24]. For instance, the canonical two-form ω can be re-interpreted as the isoform, $\hat{\omega} = \omega$, the *curvature* $\nabla = \omega\hbar^{-1}$, $\hbar = 1$, is then automatically re-interpreted as the *isocurvature* $\hat{\nabla} = \omega\hat{T}$ etc. (see ref. [24], Ch. 2 for details).

As a result, the entire formalism of symplectic quantization admits a unique and unambiguous isotopic interpretation without any major reformulation. It then follows that *hadronic mechanics of Paper II is indeed the unique and unambiguous operator image of the isohamiltonian mechanics.*These isotopies are significant for the study of nonlocal-integral and nonhamiltonian interactions in particle physics, superconductivity and other fields.

Remark 5. The nonlinear, nonlocal and noncanonical character of the isotopies is evident from the preceding analysis. It is important to point out that linearity is reconstructed in isospace and called *isolinearity,* as shown in Eq. (2.1). Locality is equally reconstructed in isospace, and called *isolocality,* because one- and two-isoforms are based on the local isodifferentials $\hat{d}\hat{x}$ and $\hat{d}\hat{p}$. Similarly, canonicity is reconstructed in isospace, and called *isocanonicity*, because the canonical form $p_k dx^k$ is preserved by the isotopic form $\hat{p}_k \hat{d}\hat{x}^k$ in isospace. The nonlinear, nonlocal and noncanonical character of isotopic theories solely emerge when they are projected in the original spaces.

Numerous other reconstruction of original properties in isospaces occur under isotopies. As an example, it is easy to see that isogroups are characterized by *nonunitary* transforms in an ordinary Hilbert space $\mathcal{H}$, i.e., for $U = \exp(i\hat{H}\hat{T}t)$, we have $UU^\dagger \neq I$ owing to the noncommutativity of $\hat{H}$ and $\hat{T}$. However, these transforms do verify the axiom of unitarity when written in the isohilbert space $\hat{\mathcal{H}}$ (Paper II). In fact, all nonunitary operators U can always be decomposed in the form $U = \hat{U}\hat{T}^{1/2}$, yielding the *isounitary law* $\hat{U}*\hat{U}^\dagger = \hat{U}\hat{T}\hat{U}^\dagger = \hat{1} = \hat{U}^\dagger * \hat{U}$.

The latter point illustrates the lack of equivalence between conventional and isotopic theories which are connected at the classical level by noncanonical transforms and at the operator level by nonunitary transforms (see [24] for details).

3. Isoriemannian geometry.

The Riemannian geometry [14] is *exactly valid* for the *exterior gravitational problem* in vacuum, because an extended body moving in the homogeneous and isotropic vacuum (such as Jupiter in its planetary trajectory around the Sun) can be effectively approximated as a massive point, thus providing the physical foundations of the local-differential character of the geometry.

As outlined in Section 1 (see [24] for details), the Riemannian geometry is only *approximately valid* for *interior gravitational problems* (such as a space-ship during re-entry in our inhomogeneous and anisotropic atmosphere) because the shape of the body considered affects its trajectory and the local-differential treatment is no longer exact.

Numerous deformations-generalizations of the Riemannian geometry have been studied during in this century to represent more general conditions, but they generally imply the abandonment of the space-time Riemannian and, therefore, of the Einsteinian axioms in favor of yet un-identified axioms.

This author submitted in 1988 [20] (see [24], Ch. 9, for a comprehensive presentation) the isotopies of the Riemannian geometry, called *isoriemannian geometry*, to achieve the desired representation of arbitrary nonlinear and nonlocal effects while preserving the original Riemannian and Einsteinian axioms. The isogeometry was constructed via *the isotopic lifting of the unit and of the conventional associative product* of the original geometry while preserving the conventional differential calculus. The emerging generalized geometry did result to be an isotopy of the original one, that is, preserving the original Riemannian axioms, while permitting the representation of nonlinear and nonlocal effects via their embedding in the generalized unit. However, the use of the conventional differential calculus implies the lack of invariance of the basic isounit, with consequential problematic aspects for measurements indicated earlier.

In this section we shall present, apparently for the first time, the isoriemannian geometry formulated via *the isotopy of the differential calculus* and show that the latter formulation is more conducive to a single, unified, abstract formulation of the geometry with different realizations, the conventional local-differential one for the exterior problem in vacuum and the more general nonlocal-integral isotopic one for interior problems within physical media. Our study will be again in local realizations representing the fixed inertial frame of the observer while all abstract treatments are left to the interested reader. For the conventional case we assume all topological properties of Lovelock and Rund [12] of which we shall preserve the symbols for clarity in the comparison of the results. For the isotopic case we assume the topological properties by Tsagas and Sourlas [30,31] implemented as per Definition 3 of Paper I which are also tacitly implied hereon. Our presentation is made, specifically, for the (3+1)-dimensional space-time, the extension to arbitrary dimensions and signatures being elementary.

Let $\Re = \Re(x,g,R)$ be a (3+1)-dimensional Riemannian space over the reals $R(n,+,\times)$ [12] with local coordinates $x = \{x^\mu\} = \{r, x^4\}$, $x^4 = c_0 t$, $\mu = 1, 2, 3, 4$, where c_0 is the speed of light in vacuum, nowhere singular, symmetric and real-valued metric $g(x) = (g_{\mu\nu})$ with tangent Minkowski

Minkowski space $M(x,\eta,R)$ with metric η = diag. (1, 1, 1, −1) .The interval the familiar expression $x^2 = x^\mu g_{\mu\nu}(x)x^\nu \in R$ with infinitesimal line element $ds^2 = dx^\mu g_{\mu\nu}(x)dx^\nu$ and related formalism (covariant derivative, Christoffel's symbols, etc. [18].

Let $\hat{\mathfrak{R}} = \hat{\mathfrak{R}}(\hat{x},\hat{g},\hat{R})$ be an isotopic image of $\mathfrak{R}$, called *isoriemannian space*, first introduced by this author in ref. [18] of 1983, with local coordinates $\hat{x} = \{\hat{x}^\mu\}$ (= $\{x^\mu\}$) and *isometric* $\hat{g} = \hat{T}g$, where $\hat{T} = (\hat{T}_\mu{}^\nu)$ is a nowhere singular, symmetric, real valued and positive-definite 4×4 matrix with C^∞ elements. The isospace $\hat{\mathfrak{R}}$ is defined over the isoreals $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ with isounit $\hat{1} = (\hat{1}^\mu{}_\nu) = \hat{T}^{-1}$. The lifting $\mathfrak{R} \to \hat{\mathfrak{R}}$ leaves unrestricted the functional dependence of the isounit/isotopic element, which can therefore depend in an arbitrarily nonlinear and nonlocal-integral way on the coordinates $\hat{x}$, velocities $\hat{v} = d\hat{x}/d\tau$, accelerations $\hat{a} = d\hat{v}/d\tau$, and any needed additional quantity of the interior medium, such as density μ, temperature τ, etc. By recalling that the original unit of $\mathfrak{R}$ is I = diag. (1, 1, 1, 1), the lifting $\mathfrak{R} \to \hat{\mathfrak{R}}$ is characterized by [23,24]

$$I = \text{diag.}(1, 1, 1, 1) \rightarrow \hat{1}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) = \hat{T}^{-1}, \quad (3.1a)$$

$$g(x) \rightarrow \hat{g}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) = \hat{T}g. \quad (3.1b)$$

We then have the *isoline element*

$$\hat{x}^2 = [\hat{x}^\mu \hat{g}_{\mu\nu}\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots)\hat{x}^\nu]\hat{1} \in \hat{R}, \quad (3.2)$$

with infinitesimal version

$$\hat{d}\hat{s}^2 = (\hat{d}\hat{x}^\alpha \hat{g}_{\alpha\beta} \hat{d}\hat{x}^\beta)\hat{1} \in \hat{R}. \quad (3.3)$$

The capability of representing arbitrarily nonlinear and nonlocal effects of the interior problem as well as inhomogeneous and anisotropic media is therefore embedded *ab initio* in the isoriemannian geometry.

The *isonormal coordinates* $\hat{y}$ occur when the isometric $\hat{g}$ is reduced, *not* to the Minkowski metric η, but rather to its isotopic image, i.e., $\hat{g}_{\hat{x}} \rightarrow \hat{\eta}_{\hat{y}} = \hat{T}_{\hat{y}}\eta_{\hat{y}}$ and, as such, they are the conventional normal coordinates (*principle of isoequivalence*). In different terms, the correct tangent space is not the conventional space $M(x,\eta,R)$, but the *isominkowskian space* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ first submitted in ref. [25]. In particular, we have the following

Lemma 3:

The isounit and related isotopic element are the same for both the isoriemannian spaces and its tangent isominkowskian spaces.

Under these conditions, the isonormal coordinates only reduce the g-component in $\hat{g} = \hat{T}g$ to the η-component of $\hat{\eta} = \hat{T}\eta$. As a result, *isonormal coordinates coincide with the conventional normal coordinates.*

It is easy to see that, despite the arbitrary functional dependence of the isometric $\hat{g}$, *all infinitely possible isotopic images $\hat{\mathfrak{R}}(\hat{x},\hat{g},\hat{R})$ of a Riemannian space $\mathfrak{R}(x,g,R)$ are locally isomorphic to the latter*, i.e., for each given metric g, $\hat{\mathfrak{R}} \approx \mathfrak{R}$ for all infinitely possible $\hat{1}$ of Kadeisvili's Class I. This is first due to the preservation by $\hat{1}$ of the axioms of I, as a result of which the field R and its isotopic image $\hat{R}$ lose any distinction at the abstract level [22]. Second, the local isomorphism $\hat{\mathfrak{R}} \approx \mathfrak{R}$ follows from the fact that, in conjunction with the deformation of the metric elements $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \hat{T}_\mu{}^\alpha g_{\alpha\nu}$, the corresponding unit has been deformed by the *inverse* amount, $I^\mu{}_\alpha \rightarrow \hat{1}^\mu{}_\alpha = (\hat{T}_\mu{}^\alpha)^{-1}$, thus preserving the original geometric characteristics. In particular, the isospace $\hat{\mathfrak{R}}$ is *isocurved*, that is (unlike the case for the isoeuclidean spaces), curvature exists in the original space and persists under isotopy.

To have an idea of the various applications under study with isoriemannian spaces, the diagonal isotopic element

$$\hat{T} = \text{diag.}(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}), \quad n_\mu > 0, \ m = 1, 2, 3, 4, \quad (3.4)$$

permits the representation of the locally varying speed $c = c_o/n_4$ of electromagnetic waves within physical media, which occurs via the fourth component of the isoline element

$$\hat{x}^4 \hat{g}_{44} \hat{x}^4 = t\, c(\hat{x}, \mu, \tau, \ldots)\, g_{44}(x)\, t, \quad c = c_o / n_4(\hat{x}, m, \tau, \ldots), \quad (3.5)$$

where g_{44} is the ordinary metric element and n_4 is the familiar index of refraction. This permits a gravitational treatment of the locally varying speed of light in interior conditions.

As an example, light propagating in our atmosphere has a dependence on the density, and then assumes yet different values when propagating in water, glasses, etc. It is evident that the representation of the locally varying speed of light is not possible with the Riemannian geometry or with its tangent Minkowskian geometry. Also, the decrease of the speed of light within inhomogeneous and anisotropic media has novel effects, such as a shift of light frequency toward the red, which cannot be predicted via the Riemannian or Minkowskian geometries, but which is quantitatively treatable in accordance with available experimental data via the isogeometries [24].

The representation technically occurs via the *isolight cone* $\hat{d}\hat{s}^2 = \hat{d}\hat{x}^\mu \hat{g}_{\mu\nu} \hat{d}\hat{x}^\nu = 0$ which is the image in isospace of the *deformed* light cone in our space-time, as generated by a locally varying speed of light. In a way similar to the fact that the isosphere is a perfect sphere in isospace (Paper I), *the isolight cone is a perfect cone in isospace* (see ref. [24], Ch. 8, for details). This occurrence is not a mere mathematical curiosity because it is important for numerical applications, such as the correct calculations of gravitational horizons. In fact, the region outside these horizon is not empty, but filled up instead by very large and hyperdense chromospheres where it is well known that the speed of light is locally varying with the density, temperature, etc., thus preventing the use of the conventional light cone. Note that the conventional exterior motion in vacuum is a particular case of the isoriemannian geometry occurring for $\hat{1} = I$.

In the first formulation of the isoriemannian geometry [20], differentials of contravariant isofields $\hat{X}^\beta$ on $\hat{\Re}$ where defined by $d\hat{X} = (\partial\hat{X}) * d\hat{x} = (\partial_\mu \hat{X})\, \hat{T}_\nu{}^\mu d\hat{x}^\nu \neq dX = (\partial_\mu X) dx^\mu$, $\partial_\mu = \partial/\partial x^\mu$ The isodifferential calculus allows us to introduce the following alternative definition

$$\hat{d}\hat{X}^\beta = (\hat{\partial}_\mu \hat{X}^\beta)\, \hat{d}\hat{x}^\mu = \hat{T}_\mu{}^\rho (\partial_\rho \hat{X}^\beta)\, \hat{1}^\mu{}_\sigma\, d\hat{x}^\sigma \equiv$$

$$\equiv (\partial_\mu \hat{X}^\beta)\, d\hat{x}^\mu , \tag{3.6}$$

namely, *isodifferential of isovector fields coincide with ordinary differentials.*

The *isocovariant differential* can be defined by

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\gamma \hat{X}^\alpha \hat{d}\hat{x}^\gamma, \tag{3.7}$$

with corresponding *isocovariant derivative*

$$\hat{X}^\beta{}_{\hat{|}\mu} = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\mu \hat{X}^\alpha , \tag{3.8}$$

where the *isochristoffel's symbols* are given by

$$\hat{\Gamma}_{\alpha\beta\gamma} = \tfrac{1}{2} (\hat{\partial}_\alpha \hat{g}_{\beta\gamma} + \hat{\partial}_\gamma \hat{g}_{\alpha\beta} - \hat{\partial}_\beta \hat{g}_{\alpha\gamma}) = \hat{\Gamma}_{\gamma\beta\alpha} , \tag{3.9a}$$

$$\hat{\Gamma}_\alpha{}^\beta{}_\gamma = \hat{g}^{\beta\rho} \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}_\gamma{}^\beta{}_\alpha , \quad \hat{g}^{\beta\rho} = [(\hat{g}_{\mu\nu})^{-1}]^{\beta\rho}, \tag{3.9b}$$

and one should note the abstract identity of the conventional and isotopic connections. The extension to covariant isofields and covariant or contravariant tensor isofields is consequential and it is hereon assumed (see also [23]).

The repetition of the proof of [2], pap. 80-81, yields to the following:

Lemma 4 (Isoricci Lemma)

Under the assumed conditions, the isocovariant derivatives of all isometrics on isoriemannian spaces are identically null,

$$\hat{g}_{\alpha\beta\hat{|}\gamma} \equiv 0, \qquad \alpha, \beta, \gamma = 1, 2, 3, 4 . \tag{3.10}$$

Despite the similarities with the conventional case, the lack of equivalence of the Riemannian and isoriemannian geometries can be illustrated via the *isotorsion* [20]

$$\hat{\tau}_\alpha{}^\beta{}_\gamma = \hat{\Gamma}_\alpha{}^\beta{}_\gamma - \hat{\Gamma}_\gamma{}^\beta{}_\alpha , \tag{3.11}$$

which is identically null for the isoriemannian geometry here considered, but its projection in the original space $\Re$ is not necessarily null. Interior gravitational models treated with the isoriemannian geometry are therefore theories with null isotorsion but generally non-null torsion as requested for a realistic treatment of *interior* problems.

The occurrence also illustrates the property, verified at subsequent levels later on, that departures from conventional geometric properties must be studied in the *projection* of isoriemannian spaces in the original spaces because, when treated in their respective spaces, the two geometries coincide. Stated in different terms, when using the conventional Riemannian geometry, exterior gravitation can only be studied in the spaces $\Re$. On the contrary, when using the isogeometry, interior gravitation can be studies in *two* different spaces, the isoriemannian spaces $\hat{\Re}$ and their projection into $\Re$.

Another way of identifying the differences between the Riemannian and isoriemannian geometries is by considering the following *isotopic Newton equations* in isoriemannian space

$$\frac{\hat{D}\hat{x}_\beta}{\hat{D}\hat{\tau}} = \frac{\partial v_\beta}{\partial \hat{\tau}} + \hat{\Gamma}_{\alpha\beta\gamma}(\hat{x}, \hat{v}, \hat{a}, \ldots) \frac{\partial \hat{x}^\alpha}{\partial \hat{\tau}} \frac{\partial \hat{x}^\gamma}{\partial \hat{\tau}} = 0, \quad (3.12)$$

where $\hat{v} = \partial\hat{x}/\partial\tau = \hat{1}^\circ_0 dx/d\tau$, $\hat{\tau}$ is the proper isotime and $\hat{1}^\circ_0$ the related isounit. The preceding equations must then be compared with the conventional equations

$$\frac{Dx_\beta}{Ds} = \frac{dv_\beta}{ds} + \Gamma_{\alpha\beta\gamma}(x) \frac{dx^\alpha}{ds} \frac{dx^\gamma}{ds} = 0. \quad (3.13)$$

It is evident that the latter equations are at most quadratic in the velocities while the isotopic equations are arbitrarily nonlinear in the velocities, as it occurs already in a flat space (Paper I). Also, the latter equations are local-differential while the former admit nonlocal-integral terms.

We now introduce: the *isocurvature tensor*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta} = \partial_\delta \hat{\Gamma}_\alpha{}^\beta{}_\gamma - \partial_\gamma \hat{\Gamma}_\alpha{}^\beta{}_\delta + \hat{\Gamma}_\rho{}^\beta{}_\delta \hat{\Gamma}_\alpha{}^\rho{}_\gamma - \hat{\Gamma}_\rho{}^\beta{}_\gamma \hat{\Gamma}_\alpha{}^\rho{}_\delta ; \quad (3.14)$$

the *isoricci tensor*

$$\hat{R}_{\mu\nu} = \hat{R}_\mu{}^\beta{}_{\nu\beta} ; \quad (3.15)$$

the *isocurvature isoscalar*

$$\hat{R} = \hat{g}^{\alpha\beta} \hat{R}_{\alpha\beta} = \hat{R}_\mu{}^\mu ; \quad (3.16)$$

the *isoeinstein tensor*

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \tfrac{1}{2} \hat{R} ; \quad (3.17)$$

and the *isotopic isoscalar*

$$\hat{\Theta} = \hat{g}^{\alpha\beta} \hat{g}^{\gamma\delta} (\hat{\Gamma}_{\rho\alpha\delta} \hat{\Gamma}_\gamma{}^\rho{}_\beta - \hat{\Gamma}_{\rho\alpha\beta} \hat{\Gamma}_\gamma{}^\rho{}_\delta) =$$

$$= \hat{\Gamma}_{\rho\alpha\beta} \hat{\Gamma}_\gamma{}^\rho{}_\delta (\hat{g}^{\alpha\delta} \hat{g}^{\gamma\beta} - \hat{g}^{\alpha\beta} \hat{g}^{\gamma\delta}). \quad (3.18)$$

the latter one being new for the isoriemannian geometry (see below).

Tedious but simple calculations then yield the following basic properties of the isoriemannian geometry:

Property 1: *Antisymmetry of the last two indices of the isocurvature tensor*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta} = -\hat{R}_\alpha{}^\beta{}_{\delta\gamma} ; \quad (3.19)$$

Property 2: *Symmetry of the first two indices of the isocurvature tensor*

$$\hat{R}_{\alpha\beta\gamma\delta} = \hat{R}_{\beta\alpha\gamma\delta} ; \quad (3.20)$$

Property 3: *Vanishing of the totally antisymmetric part of the isocurvature tensor*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta} + \hat{R}_\gamma{}^\beta{}_{\delta\alpha} + \hat{R}_\delta{}^\beta{}_{\alpha\gamma} = 0 ; \quad (3.21)$$

Property 4: *Isobianchi identity*

$$\hat{R}_\alpha{}^\beta{}_{\gamma\delta\hat{|}\rho} + \hat{R}_\alpha{}^\beta{}_{\rho\gamma\hat{|}\delta} + \hat{R}_\alpha{}^\beta{}_{\delta\rho\hat{|}\gamma} = 0 ; \quad (3.22)$$

Property 5: *Isofreud identity* (see Freud [6] for the original form, Pauli [13] for a subsequent treatment, Rund [15] for a more recent presentation and Santilli [23], Ch. 5, for a general review)

$$\hat{S}^\alpha{}_\beta = \hat{R}^\alpha{}_\beta = - \tfrac{1}{2} \delta^\alpha{}_\beta \hat{R} - \tfrac{1}{2} \delta^\alpha{}_\beta \hat{\Theta} = \hat{U}^\alpha{}_\beta + \partial_\rho \hat{V}^{\alpha\rho}{}_\beta , \quad (3.23)$$

where $\hat{\Theta}$ is the isotopic isoscalar (7.18) and

$$\hat{U}^\alpha{}_\beta = - \tfrac{1}{2} \frac{\partial \hat{\Theta}}{\partial \hat{g}^{\alpha\beta}{}_{\hat{|}\alpha}} \hat{g}^{\alpha\beta}{}_{\hat{|}\beta} , \quad (3.24a)$$

$$\hat{V}^{\alpha\rho}{}_\beta = \tfrac{1}{2} [\hat{g}^{\gamma\delta} (\delta^\alpha{}_\beta \hat{\Gamma}_\alpha{}^\rho{}_\delta - \delta^\rho{}_\beta \hat{\Gamma}_\gamma{}^\rho{}_\delta) +$$

$$+(\delta^{\rho}_{\beta}\,\hat{g}^{\alpha\gamma} - \delta^{\alpha}_{\beta}\,\hat{g}^{\rho\gamma})\,\hat{\Gamma}_{\gamma}{}^{\delta}{}_{\delta} + \hat{g}^{\rho\gamma}\,\hat{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma} - \hat{g}^{\alpha\gamma}\,\hat{\Gamma}_{\beta}{}^{\rho}{}_{\gamma}\,], \quad (3.24b)$$

Note *the abstract identity of the conventional and isotopic properties*. This confirms that the conventional and isotopic geometries can be treated at the realization-free level via one single set of axioms, as desired.

The repetition of the proof of the Theorem of [12], p. 321, leads to the following property first identified in 1988 [20] (see also [23]) and which is here recovered via the isodifferential calculus.

Theorem 2 (Fundamental Theorem for Interior Gravitation)

Under the assumed regularity and continuity conditions, the most general possible isolagrange equations $\hat{E}^{\alpha\beta} = 0$ along an actual path $\hat{P}_0$ on a (3+1)-dimensional isoriemannian space satisfying the properties:

1) Symmetry condition

$$\hat{E}^{\alpha\beta} = \hat{E}^{\beta\alpha}, \quad (3.25)$$

2) Contracted isobianchi identity

$$\hat{E}^{\alpha\beta}{}_{\hat{|}\beta} \equiv 0, \quad (3.26)$$

3) The isofreud identity

$$\hat{S}^{\alpha}{}_{\beta} = \hat{R}^{\alpha}{}_{\beta} - \tfrac{1}{2}\,\delta_{\alpha}{}^{\beta}\,\hat{R} - \tfrac{1}{2}\,\delta^{\alpha}{}_{\beta}\,\hat{\Theta} = \hat{U}^{\alpha}{}_{\beta} + \partial_{\rho}\,\hat{V}^{\alpha\rho}{}_{\beta}, \quad (3.27)$$

are given by

$$\hat{E}^{\alpha\beta} = \alpha\,\hat{g}^{\frac{1}{2}}\,(\hat{R}^{\alpha\beta} - \tfrac{1}{2}\,\hat{g}^{\alpha\beta}\,\hat{R} - \tfrac{1}{2}\,\hat{g}^{\alpha\beta}\,\hat{\Theta}) + \beta\,\hat{g}^{\alpha\beta} -$$

$$-\,\hat{g}^{\frac{1}{2}}\,\hat{D}^{\alpha\beta} = 0, \quad (3.28)$$

where $\hat{g}^{\frac{1}{2}} = (\det \hat{g})^{1/2}$, *α and β are constants and $\hat{D}^{\alpha\beta}$ is a source tensor. For α = 1 and β = 0 the interior isogravitation field equations can be written*

$$\hat{S}^{\alpha\beta} = \hat{R}^{\alpha\beta} - \tfrac{1}{2}\,\hat{g}^{\alpha\beta} - \tfrac{1}{2}\,\hat{g}^{\alpha\beta}\,\hat{\Theta} = \hat{t}^{\alpha\beta} - \hat{\tau}^{\alpha\beta} =$$

$$= \hat{U}^{\alpha}{}_{\beta} + \partial_{\rho}\,\hat{V}^{\alpha\rho}{}_{\beta}, \quad (3.29)$$

where $\hat{t}^{\alpha\beta}$ is a source tensor and $\hat{\tau}^{\alpha\beta}$ is a stress-energy tensor.

Note the appearance in Eq.s (3.29) of the isotopic isoscalar $\hat{\Theta}$ in the l.h.s and of source terms in the r.h.s., the latter ones originating from the isofreud identity. Additional studies not reported here for brevity (see 24], Ch. 9) have shown that the the tensors $\hat{t}^{\alpha\beta}$ is nowhere null and of first order in magnitude. This illustrates the principle of isoequivalence according to which under the isonormal coordinates the isometric $\hat{g}$ is indeed reduced to the tangent isominkowski metric $\hat{\eta} = T\eta$, but the source $\hat{t}^{\alpha\beta}$ cannot be rendered null.

A vector isofield $\hat{X}^{\beta}$ on $\hat{\mathfrak{R}}$ is said to be transported by *isoparallel displacement* from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\hat{\mathfrak{R}}$ to a neighboring point $\hat{m}'(\hat{x} + \hat{\partial}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\,\hat{X}^{\beta} = \hat{\partial}\,\hat{X}^{\beta} + \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma}\,\hat{X}^{\alpha}\,\hat{\partial}\hat{x}^{\gamma} \equiv 0. \quad (3.30)$$

or in integrated form

$$\hat{X}^{\beta}(\hat{m}') - \hat{X}^{\beta}(m) = \int_{\hat{m}}^{\hat{m}'} \frac{\hat{\partial}\hat{X}^{\beta}}{\hat{\partial}\hat{x}^{\alpha}}\,\frac{\hat{\partial}\hat{x}^{\alpha}}{\hat{\partial}s}\,\hat{\partial}s. \quad (3.31)$$

The isotopy of the conventional case [12] then yield the following:

Lemma 5:

Necessary and sufficient conditions for the existence of an isoparallel transport along a curve $\hat{C}$ on a (3+1)-dimensional isoriemannian space are that all the following conditions are identically verified along $\hat{C}$

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta}\,\hat{X}^{\alpha} = 0, \qquad \beta, \gamma, \delta = 1, 2, 3, 4. \quad (3.32)$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. Along similar lines, we say that a smooth path $\hat{x}_{\alpha}$ on $\hat{\mathfrak{R}}$ with isotangent $\hat{v}_{\alpha} = \hat{\partial}\hat{x}_{\alpha}/\hat{\partial}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{D}\hat{x}_{\beta}}{\hat{D}\hat{s}} = \frac{\hat{\partial}\hat{v}_{\beta}}{\hat{\partial}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma}\,\frac{\hat{\partial}\hat{x}^{\alpha}}{\hat{\partial}\hat{s}}\,\frac{\hat{\partial}\hat{x}^{\gamma}}{\hat{\partial}\hat{s}} = 0. \quad (3.33)$$

It is easy to prove the following:

Lemma 6:

The isogeodesics of an isoriemannian space $\hat{\mathfrak{R}}$ are the curves verifying the isovariational principle

$$\hat{\delta} \int^{\wedge} [\, \hat{g}_{\alpha\beta}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots)\, \hat{\partial} \hat{x}^{\alpha}\, \hat{\partial} \hat{x}^{\beta}]^{1/2} = 0. \qquad (3.34)$$

Finally, w point out the property which is inherent in the notion of isotopies as realized in this paper:

Lemma 7:

Geodesic trajectories in ordinary space remain isogeodesics in isospace.

For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the isocircle of Paper I, and the same happens for other curves. As it is the case for all other aspects, the differences between a geodesic and an isogeodesic emerge when projecting the latter in the space of the former. In fact, the projection of the isocircle in the conventional space becomes an ellipse under the assume topology (and can be a hyperbola when relaxing the positive-definite character of $\hat{1}$) [23].

We can say in figurative terms that interior physical media "disappear" under their isoriemannian geometrization, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Riemannian space) are turned into isogeodesics in isospace with the shape of the geodesics in the absence of resistive forces. This property is inherent in the very conception of the isotopic Newton equations, e.g., in representation (3.14), and it is only re-expressed in this section in an isocurved space.

In summary, a basic question raised in this section is: *why use in interior problems the Riemannian geometry with metric* g(x) *when the same axioms permit metrics* $\hat{g}(\hat{x}, \hat{v}, \hat{a}, \ldots)$ *with a more general functional dependence in the velocities and other variables as needed for interior conditions* ? In fact, at the abstract level we have the identities $I \equiv \hat{1}$, $dx \equiv \hat{\partial}\hat{x}$, $R(n,+,\times) \equiv \hat{R}(\hat{n},+,\hat{\times})$, and $\mathfrak{R}(x,g,R) \equiv \hat{\mathfrak{R}}(\hat{x},\hat{g},\hat{R})$ with consequential *unique* abstract geometric axioms for both spaces $\mathfrak{R}$ and $\hat{\mathfrak{R}}$. Within such a setting, $\mathfrak{R}$ emerges as a simpler *realization* of the Riemannian axioms, and $\hat{\mathfrak{R}}$ as a more general realization.

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