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A note on the zeros of Legendre functions

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Abstract

It is shown that real zeros of $P_v(x) = 0, -1 < x < 1$ regarded as a function of v are simple zeros.

Key words: Legendre, zeros, order.

Una nota sobre los ceros de las funciones de Legendre

Resumen

Se demuestra que los ceros reales de $P_v(x) = 0, -1 < x < 1$ considerados como función de v son ceros simples.

Palabras claves: Legendre, ceros, orden.

It is well known that the roots of $P_v(x) = 0$, (v real, $-1 < x < 1$) regarded as a function of x are simple, Hobson [1] (p.385), MacRobert [2] (p.84). What does not seem to be well known is that the zeros of $P_v(x)$ regarded as a function of real v are also simple. We shall give a simple demonstration of this fact.

Theorem

Given that v_0 is real and

$$P_{v_0}(x_0) = 0, \text{ for } -1 < x_0 < 1, -\infty < v_0 < \infty,$$

then

$$\frac{\partial P_v(x)}{\partial v} \neq 0 \text{ at } x = x_0, \; v = v_0.$$

Proof

The Legendre equation for $P_v(x)$ can be written as

$$\left[(1-x^2)P'_v\right]' + v(v+1)P_v = 0, \quad ' \equiv \frac{\partial}{\partial x}. \tag{1}$$

Differentiating this equation with respect to the parameter v gives

$$\left[(1-x^2)\left(\frac{\partial P_v}{\partial v}\right)'\right]' + v(v+1)\frac{\partial P_v}{\partial v} = -(2v+1)P_v. \tag{2}$$

Multiplying (1) by $\partial P_v/\partial v$ and (2) by P_v gives

$$\frac{\partial P_v}{\partial v}\left[(1-x^2)P'_v\right]' + v(v+1)P_v\frac{\partial P_v}{\partial v} = 0, \tag{3}$$

$$P_v\left[(1-x^2)\left(\frac{\partial P_v}{\partial v}\right)'\right]' + v(v+1)P_v\frac{\partial P_v}{\partial v} = -(2v+1)(P_v)^2, \tag{4}$$

respectively.

Subtracting equation (4) from (3) gives

$$\left[(1-x^2)\left\{\frac{\partial P_v}{\partial v}\cdot P'_v - P_v\left(\frac{\partial P_v}{\partial v}\right)'\right\}\right]' = (2v+1)(P_v)^2 \tag{5}$$

Integrating both sides of (5) from x_0 to 1 gives

$$\left[(1-x^2)\left\{\frac{\partial P_\nu(x)}{\partial \nu}P'_\nu(x) - P_\nu(x)\left(\frac{\partial P_\nu(x)}{\partial \nu}\right)'\right\}\right]_{x_0}^{1} =$$

$$(2\nu+1)\int_{x_0}^{1}\{P_\nu(x)\}^2 dx\,. \qquad (6)$$

The expression (6) reduces to $0 = 0$ for $\nu = -\frac{1}{2}$ and the equation is nugatory. To see why the left-hand side of (6) vanishes identically for $\nu = -\frac{1}{2}$ we shall use the Mehler-Dirichlet integral Hobson [1] p.26:

$$P_\nu(x) = \frac{2}{\pi}\int_0^{\cos^{-1}x}\frac{\cos(\nu+\frac{1}{2})t}{\{2(\cos t - x)\}^{\frac{1}{2}}}dt\,,\quad -1 < x < 1. \qquad (7)$$

Thus we have by legitimate differentiation with respect to ν

$$\frac{\partial P_\nu(x)}{\partial \nu} = -\frac{2}{\pi}\int_0^{\cos^{-1}x}\frac{t\sin(\nu+\frac{1}{2})t}{\{2(\cos t - x)\}^{\frac{1}{2}}}dt\,.$$

Hence we obtain the interesting result that

$$\frac{\partial P_\nu(x)}{\partial \nu}\Big|_{\nu=-\frac{1}{2}} \equiv 0\,.\ -1 < x < 1. \qquad (8)$$

Thus we shall consider the case $\nu = -\frac{1}{2}$ separately later.

We now make use of the series representation for $P_\nu(x)$, Hobson [1] p.21, p.223

$$P_\nu(x) = F(-\nu,\nu+1;1;\frac{1+x}{2})\,,\ |1-x| < 2\,;$$

$$= 1 - \frac{\nu(\nu+1)}{2}(1-x) + O((1-x)^2)\,,$$

$$\frac{\partial P_\nu(x)}{\partial \nu} = -\frac{(2\nu+1)}{2}(1-x) + O((1-x)^2)\,.$$

Thus for the upper limit of the expression on the left-hand side of the equality sign of equation (6) we have

$$(1-x^2)\left\{\frac{\partial P_\nu(x)}{\partial \nu}\frac{\partial P_\nu(x)}{\partial x} - P_\nu(x)\frac{\partial^2 P_\nu(x)}{\partial x\partial \nu}\right\} = 0\,,$$

as $x \to 1$;

and by using the fact that $P_{\nu_0}(x_0) = 0$ this reduces the expression (6) to

$$(1-x_0^2)\frac{\partial P_{\nu_0}(x_0)}{\partial \nu_0}P'_{\nu_0}(x_0) = (2\nu_0+1)\int_1^{x_0}P^2_{\nu_0}(x)dx\,.$$

Now it is well known that if $P_{\nu_0}(x_0) = 0$ then $\left.\frac{\partial P_{\nu_0}(x)}{\partial x}\right|_{x=x_0} \neq 0$. Hence we have

$$\left.\frac{\partial P_\nu(x)}{\partial \nu}\right|_{\substack{x=x_0\\ \nu=\nu_0}} = \frac{(2\nu_0+1)}{(1-x_0^2)P'_{\nu_0}(x_0)}$$

$$\int_1^{x_0}\{P_{\nu_0}(x)\}^2 dx \neq 0,\ \nu_0 \neq -\tfrac{1}{2}\,.$$

For the situation $\nu_0 = -\frac{1}{2}$ it is obvious from the integral representation

$$P_{-\frac{1}{2}}(x) = \frac{\sqrt{2}}{\pi}\int_0^{\cos^{-1}x}\frac{dt}{\sqrt{(\cos t - x)}}\,,$$

that

$$P_{-\frac{1}{2}}(x) \neq 0,\ -1 < x < 1\,.$$

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