Kober fractional q-integral operator of the basic analogue of the H-function

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Abstract

This paper deals with the derivation of the Kober fractional q-integral operator of the basic analogue of the H-function defined by Saxena, Modi and Kalla [Rev. Téc. Ing., Univ. Zulia. 6(1983), 139-143]. Several interesting results involving $G_q(.);E_q(.)$; the basic elementary functions and the basic Bessel functions such as $J_v(x; q)$; $Y_v(x; q)$; $K_v(x; q)$; $H_v(x; q)$, are deduced as the special cases of the main results.

Key words: Kober fractional q-integral operator, basic integration, basic analogue of Fox's H-function, basic Bessel functions.

Operador q-integral fraccional Kober de la análoga básica de la función H

Resumen

Este trabajo trata de la derivación del operador q-integral fraccional de Kober de la análoga básica de la función H definida por Saxena, Modi y Kalla [Rev. Téc. Ing., Univ. Zulia, 6 (1983), 139-143. Se deducen varios resultados interesantes que involucran G_q (.); E_q (.); las funciones elementales básicas y las funciones de Bessel básicas tales como $J_v(x; q)$; $Y_v(x; q)$; $K_v(x; q)$; $H_v(x; q)$, como casos especiales de los resultados principales.

Palabras clave: Operador q-integral fraccional Kober, integración básica, análoga básica de la función H de Fox, funciones de Bessel básicas.

1. Introduction

A basic analogue of the Kober fractional integral operator, as defined by Agarwal [1], is given as:

$$I_q^{\eta,\mu} f(x) = \frac{x^{-\eta-\mu}}{\Gamma_q(\mu)} \int_0^x (x - tq)_{\mu-1} t^{\eta} f(t) d(t;q) \qquad (1)$$

where μ is an arbitrary order of integration with Re(μ)> 0 and η being real or complex.

Following Agarwal [1] and Al-Salam [2], the basic integration is defined as:

$$\int_0^x f(t)d(t;q) = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k), \qquad (2)$$

By virtue of the result (2), (1) can be expressed as:

$$I_q^{\eta,\mu} f(x) = \frac{(1-q)}{\Gamma_q(\mu)} \sum_{k=0}^{\infty} q^{k(1+\eta)} \left(1-q^{k+1}\right)_{\mu-1} f\left(xq^k\right) \qquad (3)$$

where Re(μ) > 0 and η being real or complex.

Further, for real or complex α and $0< |q| < 1$, the q-shifted factorial is defined as:

$$(q^{\alpha};q)_n = \begin{cases} 1 & ; n=0 \\ (1-q^{\alpha})(1-q^{\alpha+1})\dots(1-q^{\alpha+n-1}) & ; n=1,2\dots \end{cases} \qquad (4)$$

or equivalently

$$\left(q^{\alpha};q\right)_n = \frac{\left(q^{\alpha};q\right)_\infty}{\left(q^{\alpha+n};q\right)_\infty}, \tag{5}$$

where

$$\left(q^{\alpha};q\right)_\infty = \prod_{j=0}^{\infty}\left(1-q^{\alpha+j}\right). \tag{6}$$

Also

$$\left(x-y\right)_\upsilon = x^\upsilon \prod_0^\infty \left[\frac{1-\left(y/x\right)q^n}{1-\left(y/x\right)q^{n+\upsilon}}\right], \tag{7}$$

and

$$\Gamma_q(\alpha) = \frac{\left(q;q\right)_\infty}{\left(q^{\alpha};q\right)_\infty}\left(1-q\right)^{1-\alpha} = \frac{\left(1-q\right)_{\alpha-1}}{\left(1-q\right)^{\alpha-1}}, \tag{8}$$

where $\alpha \neq 0,-1,-2...$

The q-binomial series is given by

$${}_1\phi_0\left[\begin{matrix}\alpha; \\ \quad q, x \\ -;\end{matrix}\right] = \frac{\left(\alpha x;q\right)_\infty}{\left(x;q\right)_\infty}, \tag{9}$$

Following Saxena, Modi and Kalla [3], the basic analogue of H-function is defined as:

$$H_{A,B}^{m_1,n_1}\left[x;q\middle|\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n_1} G\left(q^{1-a_j+\alpha_j s}\right)\pi x^s}{\prod_{j=m_1+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n_1+1}^{A} G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds, \tag{10}$$

where $0 \le m_1 \le B, 0 \le n_1 \le A$; α_i's and β'_js are all positive integers, the contour C is a line parallel to Re(ws)= 0, with indentations, if necessary, in such a manner that all poles of, $G\left(q^{b_j-\beta_j s}\right) 1 \le j \le m_1$ are to the right, and those of $G\left(q^{1-a_j+\alpha_j s}\right), 1 \le j \le n_1$, to the left of C. The integral converges if $\mathrm{Re}\left[s\log(x) - \log\sin\pi s\right] < 0$ for large values of $|s|$ on the contour, that is, if $\left|\left\{\arg(x) - w_2 w_1^{-1}\log|x|\right\}\right| < \pi$, where $0 < |q| < 1$, $\log q = -w = -\left(w_1 + iw_2\right)$, w, w_1, w_2 are definite quantities, w_1 and w_2 being real.

Further, the q-gamma's used above are given by

$$G\left(q^{\alpha}\right) = \left\{\prod_{n=0}^{\infty}\left(1-q^{\alpha+n}\right)\right\}^{-1} = \frac{1}{\left(q^{\alpha};q\right)_\infty}. \tag{11}$$

Indeed, it is interesting to observe that

$$\left(q;q\right)_\infty\left(1-q\right)^{1-\alpha}G\left(q^{\alpha}\right) = \Gamma_q(\alpha). \tag{12}$$

If we set $\alpha_i = \beta_j = 1$, $\forall i$ and j in equation (10), then it reduces to the basic Meijer's G-function, namely

$$H_{A,B}^{m_1,n_1}\left[x;q\middle|\begin{matrix}(a,1)\\(b,1)\end{matrix}\right] = G_{A,B}^{m_1,n_1}\left[x;q\middle|\begin{matrix}a_1,\ldots,a_A\\b_1,\ldots,b_B\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G\left(q^{b_j-s}\right)\prod_{j=1}^{n_1} G\left(q^{1-a_j+s}\right)\pi x^s}{\prod_{j=m_1+1}^{B} G\left(q^{1-b_j+s}\right)\prod_{j=n_1+1}^{A} G\left(q^{a_j-s}\right)G\left(q^{1-s}\right)\sin\pi s}ds, \tag{13}$$

where $0 \le m_1 \le B$, $0 \le n_1 \le A$ and $\mathrm{Re}\left[s\log(x) - \log\sin\pi s\right] < 0$. For the shortened notation, the basic analogue of the G-function will be denoted by $G_q(.)$. Further, if we take $n_1 = 0$; $m_1 = B$ in (13), we arrive at the following basic analogue of MacRobert's E-function

$$G_{A,B}^{B,0}\left[x;q\middle|\begin{matrix}a_1,\ldots,a_A\\b_1,\ldots,b_B\end{matrix}\right] = E_q\left[B;b_j:A;a_j:x\right]. \tag{14}$$

The basic analogues of the Bessel functions $J_\upsilon(x), Y_\upsilon(x), K_\upsilon(x), H_\upsilon(x)$; which are expressible in terms of the function $G_q(.)$ cf. Saxena and Kumar [4], are defined as:

$$J_\upsilon(x;q) = \left\{G(q)\right\}^2 G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\middle|\begin{matrix}-\\\frac{\upsilon}{2},\frac{-\upsilon}{2},1\end{matrix}\right], \tag{15}$$

where $J_\upsilon\left(x;q\right)$ denotes the q-analogue of Bessel function $J_\upsilon(x)$.

$$Y_\upsilon(x;q) = \left\{G(q)\right\}^2 G_{1,4}^{2,0}\left[\frac{x^2(1-q)^2}{4};q\middle|\begin{matrix}\frac{-\upsilon-1}{2}\\\frac{\upsilon}{2},\frac{-\upsilon}{2},\frac{-\upsilon-1}{2},1\end{matrix}\right], \tag{16}$$

where $Y_\upsilon\left(x;q\right)$ denotes the q-analogue of the Bessel function $Y_\upsilon(x)$.

$$K_\upsilon(x;q) = (1-q)G_{0,3}^{2,0}\left[\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}-\\ \frac{\upsilon}{2},\frac{-\upsilon}{2},1\end{matrix}\right.\right], \quad (17)$$

where $K_\upsilon(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_\upsilon(x)$.

$$H_\upsilon(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} G_{1,4}^{3,1}\left[\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}\frac{1+\alpha}{2}\\ \frac{\upsilon}{2},\frac{-\upsilon}{2},\frac{1+\alpha}{2},1\end{matrix}\right.\right], \quad (18)$$

where $H_\upsilon(x;q)$ is the basic analogue of the Struve function $H_\upsilon(x)$.

By virtue of the definition (13), the following elementary basic (q-) functions are expressible in terms of the basic analogue of Meijer's G-function as:

$$e_q(-x) = G(q)G_{0,2}^{1,0}\left[x(1-q);q\left|\begin{matrix}-\\ 0,1\end{matrix}\right.\right], \quad (19)$$

$$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 \times G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}-\\ \frac{1}{2},0,1\end{matrix}\right.\right] \quad (20)$$

$$\cos_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 \times G_{0,3}^{1,0}\left[\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}-\\ 0,\frac{1}{2},1\end{matrix}\right.\right] \quad (21)$$

$$\sinh_q(x) = \frac{\sqrt{\pi}}{i}(1-q)^{-1/2}\{G(q)\}^2 \times G_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}-\\ \frac{1}{2},0,1\end{matrix}\right.\right] \quad (22)$$

$$\cosh_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 \times G_{0,3}^{1,0}\left[-\frac{x^2(1-q)^2}{4};q\left|\begin{matrix}-\\ 0,\frac{1}{2},1\end{matrix}\right.\right]. \quad (23)$$

A detailed account of various special functions expressible in terms of Meijer's G-function [5] or Fox's H-function can be found in research monographs by Mathai and Saxena [6, 7].

The object of this paper is to evaluate Kober fractional basic integral operator of the basic analogue of the H-function and then to deduce certain results involving $G_q(.)$, $E_q(.)$, various basic Bessel and q-elementary functions as applications of the main result. The results obtained in this paper are believed to be a new contribution to the theory of q-fractional calculus and are likely to find certain applications to the solution of the fractional q-integral equations involving various q-hypergeometric functions. In this connection one can refer to the work of Prabhakar and Chakrabarty [8], and Yadav and Purohit [9].

2. Main Results

In this section, we will evaluate the following fractional q-integral operator of Kober type involving basic analogue of the H-function.

$$I_q^{\eta,\mu}\left\{H_{A,B}^{m_1,n_1}\left[\rho x^\lambda;q\left|\begin{matrix}(a,\alpha)\\ (b,\beta)\end{matrix}\right.\right]\right\} = (1-q)^\mu \times H_{A+1,B+1}^{m_1,n_1+1}\left[\rho x^\lambda;q\left|\begin{matrix}(-\eta,\lambda),(a,\alpha)\\ (b,\beta),(-\mu-\eta,\lambda)\end{matrix}\right.\right], \lambda \geq 0 \quad (24)$$

$$I_q^{\eta,\mu}\left\{H_{A,B}^{m_1,n_1}\left[\rho x^\lambda;q\left|\begin{matrix}(a,\alpha)\\ (b,\beta)\end{matrix}\right.\right]\right\} = (1-q)^\mu \times H_{A+1,B+1}^{m_1+1,n_1}\left[\rho x^\lambda;q\left|\begin{matrix}(a,\alpha),(1+\eta+\mu,-\lambda)\\ (1+\eta,-\lambda),(b,\beta)\end{matrix}\right.\right], \lambda < 0 \quad (25)$$

where $0 \leq m_1 \leq B$, $0 \leq n_1 \leq A$ and $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0, 0 < |q| < 1$.

Proof: Using (3) and (10) the left hand side of (24) can be expressed as:

$$(1-q)^\mu \sum_{k=0}^{\infty} \frac{q^{k(1+\eta)}(q^\mu;q)_k}{(q;q)_k}\frac{1}{2\pi i} \times \int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s}) \prod_{j=1}^{n_1} G(q^{1-\alpha_j+\alpha_j s})\pi(\rho x^\lambda q^{\lambda k})^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s}) \prod_{j=n_1+1}^{A} G(q^{\alpha_j-\alpha_j s})G(q^{1-s})\sin\pi s} ds,$$

where we use the formula [10]

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}$$

with α any complex number.

On interchanging the order of summation and integration, which is valid under conditions given with (24), we obtain

$$\frac{(1-q)^{\mu}}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n_1} G\left(q^{1-a_j+\alpha_j s}\right)}{\prod_{j=m_1+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n_1+1}^{A} G\left(q^{a_j-\alpha_j s}\right)}\times \frac{\pi\left(\rho x^{\lambda}\right)^{s}}{G\left(q^{1-s}\right)\sin\pi s}\sum_{k=0}^{\infty}\frac{q^{k(1+\eta+\lambda s)}\left(q^{\mu};q\right)_k}{(q;q)_k}ds.$$

On summing the inner ${}_1\phi_0(\cdot)$ series with the help of equation (9) and then making use of the relation (5), it yields

$$\frac{(1-q)^{\mu}}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n_1} G\left(q^{1-a_j+\alpha_j s}\right)}{\prod_{j=m_1+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)G\left(q^{1+\eta+\mu+\lambda s}\right)}\times \frac{G\left(q^{1+\eta+\lambda s}\right)\pi\left(\rho x^{\lambda}\right)^{s}}{\prod_{j=n_1+1}^{A} G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\pi s}ds.$$

The above expression on being interpreted with the help of (10), yields the right hand side of (24).

Similarly, we can prove (25).

3. Applications

In this section, we will deduce the following eleven results given in tabular form, as applications, of the general result (24).

The results (26) and (27) follows directly from (24) in view of the definitions (13) and (14) respectively. Where as the results (28)-(31) can be obtained by taking $\rho = \frac{(1-q)^2}{4}$ and $\lambda = 2$ in (24) and make use of the results (15)-(18). Similarly, one can prove the results (32)-(36) by suitably assigning values to the parameters in light of the definitions (19)-(23). It is further interesting to observe that for $\eta = 0$, and on employing the relation

$$I_q^{\mu} f(x) = x^{\mu} I_q^{0,\mu} f(x) \tag{37}$$

one can obtain several special cases, where $I_q^{\mu} f(x)$ denotes the Riemann-Liouville fractional q-integral operator, given by

$$I_q^{\mu} f(x) = \frac{1}{\Gamma_q(\mu)}\int_0^x (x-yq)_{\mu-1} f(y)\,d(y;q) \tag{38}$$

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Table 1

Eq. No	$f(x)$	$I_q^{\eta,\mu} f(x) = \frac{x^{-\eta-\mu}}{\Gamma_q(\mu)}\int_0^x (x-tq)_{\mu-1} t^{\eta} f(t)\,d(t;q)$ $\mathrm{Re}(\mu) > 0$ and for all η
26	$G_{A,B}^{m_1,n_1}\left[x;q\,\middle\vert\,\begin{matrix}a_1,\dots,a_A\\ b_1,\dots,b_B\end{matrix}\right]$	$(1-q)^{\mu} G_{A+1,B+1}^{m_1,n_1+1}\left[x;q\,\middle\vert\,\begin{matrix}-\eta,a_1,\dots,a_A\\ b_1,\dots,b_B,-\mu-\eta\end{matrix}\right]$ $0 \le m_1 \le B, 0 \le n_1 \le A$ and $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0$
27	$Eq[B;b_j:A;a_j:x]$	$(1-q)^{\mu} G_{A+1,B+1}^{B,1}\left[x;q\,\middle\vert\,\begin{matrix}-\eta,a_1,\dots,a_A\\ b_1,\dots,b_B,-\mu-\eta\end{matrix}\right]$
28	$J_{\upsilon}(x;q)$	$(1-q)^{\mu}\{G(q)\}^2 H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\,\middle\vert\,\begin{matrix}(-\eta,2)\\ \left(\frac{\upsilon}{2},1\right), \left(\frac{-\upsilon}{2},1\right), (1,1), (-\mu-\eta,2)\end{matrix}\right]$
29	$Y_{\upsilon}(x;q)$	$(1-q)^{\mu}\{G(q)\}^2 H_{2,5}^{2,1}\left[\frac{x^2(1-q)^2}{4};q\,\middle\vert\,\begin{matrix}(-\eta,2),\left(\frac{-\upsilon-1}{2},1\right)\\ \left(\frac{\upsilon}{2},1\right), \left(\frac{-\upsilon}{2},1\right),\left(\frac{-\upsilon-1}{2},1\right), (1,1), (-\mu-\eta,2)\end{matrix}\right]$
30	$K_{\upsilon}(x;q)$	$(1-q)^{\mu+1} H_{1,4}^{2,1}\left[\frac{x^2(1-q)^2}{4};q\,\middle\vert\,\begin{matrix}(-\eta,2)\\ \left(\frac{\upsilon}{2},1\right), \left(\frac{-\upsilon}{2},1\right),(1,1), (-\mu-\eta,2)\end{matrix}\right]$

Table 1
(Continuation)

31	$H_{\upsilon}(x;q)$	$(1-q)^{\mu+1-\alpha}\, 2^{\alpha-1} H_{2,5}^{3,2}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\eta,2),\left(\frac{\alpha+1}{2},1\right)\\ \left(\frac{\upsilon}{2},1\right),\ \left(\frac{-\upsilon}{2},1\right),\left(\frac{\alpha+1}{2},1\right),\ (1,1),\ (-\mu-\eta,2)\end{matrix}\right]$
32	$e_q(-x)$	$(1-q)^{\mu} G(q) G_{1,3}^{1,1}\left[x(1-q);q\middle\vert\begin{matrix}-\eta\\ 0,1,-\mu-\eta\end{matrix}\right]$
33	$\sin_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\eta,2)\\ \left(\frac{1}{2},1\right),\ (0,1),(1,1),\ (-\mu-\eta,2)\end{matrix}\right]$
34	$\cos_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\eta,2)\\ (0,1),\ \left(\frac{1}{2},1\right),(1,1),\ (-\mu-\eta,2)\end{matrix}\right]$
35	$\sinh_q(x)$	$\frac{\sqrt{\pi}}{i}(1-q)^{\mu-1/2}\{G(q)\}^2 H_{1,4}^{1,1}\left[\frac{-x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\eta,2)\\ \left(\frac{1}{2},1\right),\ (0,1),(1,1),\ (-\mu-\eta,2)\end{matrix}\right]$
36	$\cosh_q(x)$	$\sqrt{\pi}(1-q)^{\mu-1/2}\{G(q)\}^2 H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\eta,2)\\ (0,1),\ \left(\frac{1}{2},1\right),(1,1),\ (-\mu-\eta,2)\end{matrix}\right]$

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